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On the free boundary problem to the two viscous immiscible fluids

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ABSTRACT

In this paper, we prove the local solvability of the free boundary problem describing the motion of two layers of immiscible, heavy, viscous, incompressible fluid lying above an infinite rigid bottom and with surface tension on the interfaces, and global solvability near the equilibrium state.

1. Introduction

The mathematical study to the interface between two layers of immiscible fluid of different densities has attracted interest because of its strong physical background (like [4] for the large-amplitude internal waves in different oceans), the challenging modelings, mathematical and numerical issues that arise in the analysis of this system. The aim of this paper is to study the problem of describing the motion of two layers of immiscible, heavy, viscous, incompressible fluid lying above an infinite rigid bottom and with surface tension on the interfaces. Before the mathematical presentation of our problem, we first give a short survey concerning the free boundary of the incompressible Navier–Stokes equations.

In the pioneering papers [11–15,17], Solonnikov studied the motion of finite isolated mass of an incompressible viscous fluid. He first proved the local solvability in a Hölder space in [11] and global solvability in the space $W_p^{2,1}$ for $p > d$ in [13] without surface tension and he obtained the local existence in [12,14,15] and global existence in [17] with surface tension. A similar problem describing the motion of one layer of fluid having a non-compact free interface with no surface tension was studied by Beale in [2,3]. He solved the problem in the Sobolev–Slobodetski space $W_2^{r,r/2}$ with $r \in (7/2, 4)$

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by means of different framework rather than solving the system in Lagrangian coordinates. Tani and Tanaka [20], Tani [19] considered the same problem but with surface tension on the free interface in the space $W_2^{r,r/2}$ with $r \in (5/2, 3)$. Their frameworks are similar to Solonnikov (see [15,17]).

While in [21,22], Tanaka studied the solvability of two phase free boundary problem in the following setting: let $\Omega^{(1)}$ and $\Omega^{(2)}$ be bounded domains in $\mathbb{R}^3$ filled with immiscible fluids (1) and (2) at the initial moment whose boundaries consist of the interface Γ between fluids (1) and (2) and fixed Σ such that $\partial\Omega^{(1)} = \Gamma$, $\partial\Omega^{(2)} = \Gamma \cup \Sigma$, $\Gamma \cap \Sigma = \emptyset$. Subject to surface tension on the free interface, he first proved the local solvability of the problem with general data [21] and global solvability for data close enough to equilibrium state [22]. Very recently, a similar problem for Euler system was studied by Shatah and Zeng [9].

On the other hand, there are lots of progresses on the free boundary problem for non-viscous fluid. In the celebrated papers [23,24], Wu proved the local solvability for the free boundary problem of curl-free Euler system but without surface tension on the free surface. When the initial vorticity does not equal 0, the solvability to the free boundary problem of Euler system was thoroughly studied by authors in [5,7,8,25,26].

Let us formulate our problem. Given smooth enough functions $b, F_0^{(1)}, F_0^{(2)}$ on $\mathbb{R}^2$, we denote $\Omega^{(1)} \stackrel{\text{def}}{=} \{x = (x', x_3) \in \mathbb{R}^3 \mid x' = (x_1, x_2) \in \mathbb{R}^2,\ F_0^{(2)}(x') < x_3 < F_0^{(1)}(x')\}$ and $\Omega^{(2)} \stackrel{\text{def}}{=} \{x \in \mathbb{R}^3 \mid x' \in \mathbb{R}^2,$ $b(x') < x_3 < F_0^{(2)}(x')\}$ be two subdomains in $\mathbb{R}^3$ filled with immiscible fluids (1) and (2) at the initial moment. Let $S_F^{(1)} \stackrel{\text{def}}{=} \{x \in \mathbb{R}^3 \mid x' \in \mathbb{R}^2,\ x_3 = F_0^{(1)}(x')\}$ be the upper boundary of $\Omega^{(1)}$, $S_F^{(2)} \stackrel{\text{def}}{=} \{x \in \mathbb{R}^3 \mid x' \in \mathbb{R}^2,\ x_3 = F_0^{(2)}(x')\}$ be the interface between fluids (1) and (2), and $S_B \stackrel{\text{def}}{=} \{x \in \mathbb{R}^3 \mid x' \in \mathbb{R}^2,$ $x_3 = b(x')\}$ be the bottom of $\Omega^{(2)}$. Then given external force $f^{(j)}$, $j = 1, 2$, for $t > 0$, we want to find domains

$$\Omega^{(1)}(t) = \left\{x \in \mathbb{R}^3 \;\middle|\; x' \in \mathbb{R}^2,\ F^{(2)}\left(x', t\right) < x_3 < F^{(1)}\left(x', t\right)\right\}$$

and

$$\Omega^{(2)}(t) = \left\{x \in \mathbb{R}^3 \;\middle|\; x' \in \mathbb{R}^2,\ b\left(x'\right) < x_3 < F^{(2)}\left(x', t\right)\right\}$$

occupied by fluids (1) and (2) respectively, and fluid densities $\rho^{(1)}, \rho^{(2)}$, fluid velocities $v^{(1)}, v^{(2)}$, pressure functions $p^{(1)}, p^{(2)}$, such that for $j = 1, 2$,

$$\begin{cases} \left(\dfrac{D}{Dt}\right)^{(j)} \rho^{(j)} = 0, \quad x \in \Omega^{(j)}(t),\ t > 0, \\ \rho^{(j)}\left(\dfrac{D}{Dt}\right)^{(j)} v^{(j)} - \nu^{(j)}\nabla^2 v^{(j)} + \nabla p^{(j)} = \rho^{(j)} f^{(j)}, \\ \nabla \cdot v^{(j)} = 0, \\ v^{(j)}\big|_{t=0} = v_0^{(j)}(x), \quad \rho^{(j)}\big|_{t=0} = \rho_0^{(j)}(x), \quad x \in \Omega^{(j)}, \end{cases} \tag{1.1}$$

together with the boundary conditions:

$$\begin{cases} T^{(1)} n^{(1)} = \sigma^{(1)} H^{(1)} n^{(1)} - g_0\left(x_3 - h^{(1)}\right) n^{(1)}, \quad x \in S_F^{(1)}(t),\ t > 0, \\ v^{(1)} - v^{(2)} = 0, \quad x \in S_F^{(2)}(t),\ t > 0, \\ \left(T^{(2)} - T^{(1)}\right) n^{(2)} = \sigma^{(2)} H^{(2)} n^{(2)} - g_0\left(x_3 - h^{(2)}\right) n^{(2)}, \quad x \in S_F^{(2)}(t),\ t > 0, \\ v^{(2)} = 0, \quad x \in S_B,\ t > 0. \end{cases} \tag{1.2}$$

Here and in what follows, we denote $(\frac{D}{Dt})^{(j)} \stackrel{\text{def}}{=} \frac{\partial}{\partial t} + v^{(j)} \cdot \nabla$ the material derivative, $\nabla = \nabla_x = {}^t(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$; $T^{(j)} \stackrel{\text{def}}{=} \nu^{(j)}\mathrm{D}(v^{(j)}) - p^{(j)}I$ the stress tensor, $\mathrm{D}(v) \stackrel{\text{def}}{=} \nabla v + {}^t(\nabla v)$ twice the velocity deformation tensor and I the 3×3 unit matrix, $n^{(1)} = n^{(1)}(x,t)$ the unit outward normal to $S_F^{(1)}(t)$, $n^{(2)} = n^{(2)}(x,t)$ the unit normal from $\Omega^{(2)}(t)$ pointing to $\Omega^{(1)}(t)$ on $S_F^{(2)}(t)$, g_0 (>0) the acceleration of gravity. And the real pressure $\tilde{p}^{(j)}$ is given by $\tilde{p}^{(j)} = p_e - g_0(x_3 - h^{(j)}) + p^{(j)}$ with the atmospheric pressure p_e above the fluid (1). While $h^{(j)}$ is a constant $\nu^{(j)}$ (>0) is the constant viscosity coefficient of fluid (j); $\sigma^{(j)}$ (>0) is the constant coefficient of surface tension on $S_F^{(j)}(t)$; $H^{(j)}/2 = H^{(j)}(x,t)/2$ is the mean curvature of the free surface $S_F^{(j)}(t)$, which is supposed to be negative when $\Omega^{(j)}(t)$ is convex in a neighborhood of x so that

$$H^{(j)}n^{(j)} = \Delta^{(j)}(t)x,$$

where $\Delta^{(j)}(t)$ is the Laplace–Beltrami operator on $S_F^{(j)}(t)$. In particular, if (s_1, s_2) is the local coordinate on a surface $S_F'^{(j)}(t) \subseteq S_F^{(j)}(t)$, so that $S_F'^{(j)}(t)$ is given by the equation $x = \vec{r}(s_1, s_2)$, then

$$\Delta^{(j)}(t) = \frac{1}{\sqrt{g}} \sum_{\alpha,\beta=1}^{2} \frac{\partial}{\partial s_\alpha}\left(\sqrt{g}\, g^{\alpha\beta} \frac{\partial}{\partial s_\beta}\right), \tag{1.3}$$

with

$$g \stackrel{\text{def}}{=} \det(g_{\alpha\beta})_{\alpha,\beta=1,2}, \qquad g_{\alpha\beta} = \frac{\partial \vec{r}}{\partial s_\alpha} \cdot \frac{\partial \vec{r}}{\partial s_\beta}, \qquad \left(g^{\alpha\beta}\right)_{\alpha,\beta=1,2} = (g_{\alpha\beta})^{-1}_{\alpha,\beta=1,2}.$$

Aside from the dynamical conditions on $S_F^{(j)}(t)$, a kinetic condition is imposed on $S_F^{(j)}(t)$ as well:

$$\left(\frac{D}{Dt}\right)^{(j)} G^{(j)} = 0 \quad \text{on } S_F^{(j)}(t), \tag{1.4}$$

if $S_F^{(j)}(t)$ is described by the equation $G^{(j)}(x,t) \equiv 0$, $j = 1, 2$.

As usual in the study of the free boundary problem, we are going to use Lagrangian coordinate to transform the free boundary problem (1.1)–(1.2) to a problem with a fixed boundary. In fact, given smooth enough velocities $v^{(j)}$, we can introduce the characteristic transformation $X^{(j)}$ by

$$\frac{d}{d\tau} X^{(j)}(\tau; \xi, t) = v^{(j)}\big(X^{(j)}(\tau; \xi, t), \tau\big), \qquad X^{(j)}(t; \xi, t) = \xi \quad \text{for } 0 \leqslant \tau \leqslant t, \tag{1.5}$$

and we denote

$$u^{(j)}(\xi, t) \stackrel{\text{def}}{=} v^{(j)}\big(X^{(j)}(t; \xi, 0), t\big),$$
$$X_{u^{(j)}}(\xi, t) \stackrel{\text{def}}{=} X^{(j)}(t; \xi, 0) = \xi + \int_0^t u^{(j)}(\xi, \tau)\, d\tau.$$

Then thanks to (1.2) and (1.4), for each $t > 0$, $X_{u^{(j)}}(\cdot, t)$ is a one-to-one mapping from $\Omega^{(j)}$ to $\Omega^{(j)}(t)$. Furthermore, there hold

$$n^{(1)}_{u^{(1)}}(\xi,t)=n^{(1)}\big(X_{u^{(1)}}(\xi,t),t\big)=\frac{\mathcal{A}_{u^{(1)}}N^{(1)}}{|\mathcal{A}_{u^{(1)}}N^{(1)}|},$$
$$n^{(2)}_{u^{(j)}}(\xi,t)=n^{(2)}\big(X_{u^{(j)}}(\xi,t),t\big)=\frac{\mathcal{A}_{u^{(j)}}N^{(2)}}{|\mathcal{A}_{u^{(j)}}N^{(2)}|},\quad j=1,2, \tag{1.6}$$

where $\mathcal{A}_{u^{(j)}}\overset{\text{def}}{=}{}^t(\frac{\partial X_{u^{(j)}}}{\partial\xi})^*={}^t(\frac{\partial X_{u^{(j)}}}{\partial\xi})^{-1}$ the adjugate matrix of ${}^t(\frac{\partial X_{u^{(j)}}}{\partial\xi})$ (here $\mathrm{Det}((\frac{\partial X_{u^{(j)}}}{\partial\xi}))=1$) and $N^{(j)}=n^{(j)}|_{t=0}$, $j=1,2$. And we can reformulate the free boundary problem (1.1)–(1.2) to the following problem with fixed boundary:

$$\rho^{(j)}\big(X_{u^{(j)}}(\xi,t),t\big)=\rho_0^{(j)}(\xi),\quad j=1,2, \tag{1.7}$$

and

$$\begin{cases}\rho_0^{(j)}u_t^{(j)}-\nu^{(j)}\nabla^2_{u^{(j)}}u^{(j)}+\nabla_{u^{(j)}}q^{(j)}=\rho_0^{(j)}f^{(j)}\big(X_{u^{(j)}}(\xi,t),t\big),\quad \xi\in\Omega^{(j)},\ t>0,\\ \nabla_{u^{(j)}}\cdot u^{(j)}=0,\quad \xi\in\Omega^{(j)},\ t>0,\\ u^{(j)}\big|_{t=0}=v_0^{(j)}(\xi),\quad \xi\in\Omega^{(j)},\ j=1,2,\end{cases} \tag{1.8}$$

together with the boundary conditions:

$$\begin{cases}\Pi^{(1)}_{u^{(1)}}\big[\nu^{(1)}\mathrm{D}_{u^{(1)}}\big(u^{(1)}\big)n^{(1)}_{u^{(1)}}\big]=0,\quad \xi\in S_F^{(1)},\ t>0,\\ -q^{(1)}+2\nu^{(1)}n^{(1)}_{u^{(1)}}\cdot\big(\nabla_{u^{(1)}}u^{(1)}n^{(1)}_{u^{(1)}}\big)\\ \quad=\sigma^{(1)}n^{(1)}_{u^{(1)}}\cdot\Delta^{(1)}_{u^{(1)}}(t)X_{u^{(1)}}-g_0\big(X_{u^{(1)},3}-h^{(1)}\big),\quad \xi\in S_F^{(1)},\ t>0,\\ u^{(1)}-u^{(2)}=0,\quad \xi\in S_F^{(2)},\ t>0,\\ \nu^{(2)}\Pi^{(2)}_{u^{(2)}}\big[\mathrm{D}_{u^{(2)}}\big(u^{(2)}\big)n^{(2)}_{u^{(2)}}\big]-\nu^{(1)}\Pi^{(2)}_{u^{(1)}}\big[\mathrm{D}_{u^{(1)}}\big(u^{(1)}\big)n^{(2)}_{u^{(1)}}\big]=0,\quad \xi\in S_F^{(2)},\ t>0,\\ q^{(1)}-q^{(2)}+2\nu^{(2)}n^{(2)}_{u^{(2)}}\cdot\big(\nabla_{u^{(2)}}u^{(2)}n^{(2)}_{u^{(2)}}\big)-2\nu^{(1)}n^{(2)}_{u^{(1)}}\cdot\big(\nabla_{u^{(1)}}u^{(1)}n^{(2)}_{u^{(1)}}\big)\\ \quad=\dfrac{\sigma^{(2)}}{2}\displaystyle\sum_{j=1}^{2}n^{(2)}_{u^{(j)}}\cdot\Delta^{(2)}_{u^{(j)}}(t)X_{u^{(j)}}-g_0\big(X_{u^{(2)},3}-h^{(2)}\big),\quad \xi\in S_F^{(2)},\ t>0,\\ u^{(2)}=0,\quad \xi\in S_B,\ t>0,\end{cases} \tag{1.9}$$

where $q^{(j)}(\xi,t)=p^{(j)}(X_{u^{(j)}}(\xi,t),t)$, $\nabla_{u^{(j)}}=\mathcal{A}_{u^{(j)}}\nabla_\xi$, $\nabla_\xi={}^t(\frac{\partial}{\partial\xi_1},\frac{\partial}{\partial\xi_2},\frac{\partial}{\partial\xi_3})$,

$$\Pi_u^{(j)}f=f-\big(f\cdot n_u^{(j)}\big)n_u^{(j)},\qquad \mathrm{D}_{u^{(j)}}(u)=\nabla_{u^{(j)}}u+{}^t(\nabla_{u^{(j)}}u),\quad j=1,2,$$

and $\Delta^{(k)}_{u^{(j)}}(t)$ is the Laplace–Beltrami operator on $S_F^{(k)}(t)$ parameterized by ξ with $X_{u^{(j)}}$ $(j,k=1,2)$.

Now we present the main results of this paper:

Theorem 1.1 *(Local existence). Let $l\in(1/2,1)$, $\sigma^{(j)}>0$, $\nu^{(j)}>0$, $h^{(j)}$ be a constant, $F_0^{(j)}-h^{(j)}\in W_2^{l+5/2}(\mathbb{R}^2)$ i.e. $S_F^{(j)}\in W_2^{l+5/2}$, $j=1,2$. Assume that $\nabla\rho_0^{(j)}\in W_2^l(\Omega^{(j)})$ with $0<M_1^{(j)}\leqslant\rho_0^{(j)}\leqslant M_2^{(j)}<+\infty$, that $f^{(j)}\in W_2^{l+4,l/2+2}(\mathbb{R}^3_\infty)$ and $v_0^{(j)}\in W_2^{l+1}(\Omega^{(j)})$ satisfy the compatibility conditions*:

$$\begin{cases} \nabla \cdot v_0^{(j)} = 0 \quad in\ \Omega^{(j)},\ j = 1,2, \\ \Pi^{(1)}\big[\nu^{(1)}\mathrm{D}\big(v_0^{(1)}\big)N^{(1)}\big] = 0 \quad on\ S_F^{(1)}, \quad v_0^{(1)} - v_0^{(2)} = 0 \quad on\ S_F^{(2)}, \\ \Pi^{(2)}\big[\nu^{(2)}\mathrm{D}\big(v_0^{(2)}\big)N^{(2)} - \nu^{(1)}\mathrm{D}\big(v_0^{(1)}\big)N^{(2)}\big] = 0 \quad on\ S_F^{(2)}, \\ v_0^{(2)} = 0 \quad on\ S_B, \end{cases}$$

where $\Pi^{(j)} f = f - (f \cdot N^{(j)})N^{(j)}$, $j = 1,2$. *Then* (1.8)–(1.9) *has a unique solution* $(u^{(1)}, q^{(1)}, u^{(2)}, q^{(2)})$ *with* $(u^{(j)}, q^{(j)}, \nabla q^{(j)}) \in W_2^{l+2,l/2+1}(Q_T^{(j)}) \times W_2^{l,l/2}(Q_T^{(j)}) \times W_2^{l,l/2}(Q_T^{(j)})$, $j = 1,2$, *for some* $T \in (0,+\infty)$ *such that* $q^{(1)}|_{S_F^{(1)}} \in W_2^{l+1/2,l/2+1/4}(S_{F,T}^{(1)})$ *and* $q^{(1)} - q^{(2)}|_{S_F^{(2)}} \in W_2^{l+1/2,l/2+1/4}(S_{F,T}^{(2)})$ *and*

$$\begin{aligned} E(0,T) &\stackrel{\mathrm{def}}{=} \sum_{j=1}^{2}\big(\|u^{(j)}\|_{W_2^{l+2,l/2+1}(Q_T^{(j)})} + \|\nabla q^{(j)}\|_{W_2^{l,l/2}(Q_T^{(j)})} + \|q^{(j)}\|_{W_2^{l,l/2}(Q_T^{(j)})}\big) \\ &\quad + \|q^{(1)}\|_{W_2^{l+1/2,l/2+1/4}(S_{F,T}^{(1)})} + \|q^{(1)} - q^{(2)}\|_{W_2^{l+1/2,l/2+1/4}(S_{F,T}^{(2)})} \\ &\leqslant C\sum_{j=1}^{2}\big(\|f^{(j)}\|_{W_2^{l+4,l/2+2}(\mathbb{R}_\infty^3)} + \|v_0^{(j)}\|_{W_2^{l+1}(\Omega^{(j)})} + \sigma^{(j)}\|F_0^{(j)} - h^{(j)}\|_{W_2^{l+5/2}(\mathbb{R}^2)}\big) \\ &\stackrel{\mathrm{def}}{=} CE_0^*. \end{aligned} \tag{1.10}$$

Here $S_{F,T}^{(j)} = \Omega^{(j)} \times (0,T)$, $Q_T^{(j)} = \Omega^{(j)} \times (0,T)$, *and* C *is a uniform constant independent of* T *and* $T \to +\infty$ *as* $E_0^* \to 0$.

Remark 1.1. In [21], Tanaka obtained a similar result in $W_2^{r,r/2}$ with $r \in (7/2, 4)$ about the two-phase free boundary problem in the compact domain with only one free interface. Therein the densities were constant and the effect of the temperature was considered. In this paper, we consider two-phase problem of inhomogeneous Navier–Stokes equations with two different free surfaces in the non-compact domain. Thus, we need to solve the linearized system with more complicated boundary conditions. We seek the solution of the linearized system by dividing it into two parts that one without surface tension and the other with surface tension. The proofs depend on solving the localized problem in the small domain. But since the domain is not compact, the covers of the domain are not finite. So we have to use subtle tricks to deal with this problem. On the other hand, since the fluids are inhomogeneous, the densities are not constants. Fortunately, the density can be exactly solved by using Lagrangian coordinates, and the trouble is that the coefficients of the linearized system depend on the density. This difficulty can be overcome by fixing the density in localized system and the fixed point argument.

Theorem 1.2 *(Global existence). Under the assumption of Theorem* 1.1, *if* $E_0 \leqslant \epsilon$ *with* $\epsilon \ll 1$, *then the solution to* (1.7)–(1.9) *with* $f^{(j)} = 0$ *exists for all* $t > 0$ *and satisfies*

$$\begin{aligned} &\sup_{t \geqslant t_1}\sum_{j=1}^{2}\big(\|v^{(j)}\|_{W_2^{l+2}(\Omega^{(j)}(t))} + \|v_t^{(j)}\|_{W_2^{l}(\Omega^{(j)}(t))} + \|p^{(j)}\|_{W_2^{l+1}(\Omega^{(j)}(t))} \\ &\quad + \sigma^{(j)}\|F^{(j)} - h^{(j)}\|_{W_2^{l+5/2}(\mathbb{R}^2)} + \|\nabla\rho^{(j)}\|_{W_2^{l}(\Omega^{(j)}(t))}\big) \leqslant C_0(t_1)E_0 \end{aligned}$$

for each $t_1 > 0$. Here $C_0(t_1)$ is a constant depending on t_0, and

$$E_0 \overset{\text{def}}{=} \sum_{j=1}^{2} \left(\left\| v_0^{(j)} \right\|_{W_2^{l+1}(\Omega^{(j)})} + \sigma^{(j)} \left\| F_0^{(j)} - h^{(j)} \right\|_{W_2^{l+5/2}(\mathbb{R}^2)} + \left\| \nabla \rho_0^{(j)} \right\|_{W_2^{l}(\Omega^{(j)})} \right).$$

Remark 1.2. A similar result was obtained by Tanaka in the same spaces in [22]. He studied the system in the bounded domain with only one free boundary. And the external force $f \in W_2^{5+l,5/2+l/2}(\mathbb{R}^3_\infty)$ was considered there, while in this paper we set $f = 0$. But we can also obtain the same result with external force.

Throughout this paper, the constant C doesn't depend on the parameters λ, γ and T which may be different from line to line. Otherwise, we will indicate this with $C(\cdots)$.

2. Functional spaces

For the readers' convenience, we first recall the definitions of Sobolev–Slobodetski spaces from [1, 10,15]. Let Ω be a subdomain of $\mathbb{R}^n$ with smooth boundary Γ. We define by $W_2^r(\Omega)$ the set of functions on Ω with finite norm

$$\|u\|_{W_2^r(\Omega)} \overset{\text{def}}{=} \left(\sum_{|\alpha|<r} \left\| D^\alpha u \right\|^2_{L^2(\Omega)} + \|u\|^2_{\dot{W}_2^r(\Omega)} \right)^{1/2}$$

and

$$\|u\|^2_{\dot{W}_2^r(\Omega)} \overset{\text{def}}{=} \begin{cases} \sum_{|\alpha|=r} \|D^\alpha u\|^2_{L^2(\Omega)} & \text{if } r \text{ is an integer,} \\ \sum_{|\alpha|=[r]} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2(r-[r])}}\, dx\, dy & \text{otherwise,} \end{cases}$$

where $[r]$ denotes the integral part of r.

Let $Q_T = \Omega \times (0,T)$, we define by $W_2^{r,r/2}(Q_T)$ the set of the functions on Q_T with finite norm

$$\|u\|^2_{W_2^{r,r/2}(Q_T)} \overset{\text{def}}{=} \|u\|^2_{W_2^{r,0}(Q_T)} + \|u\|^2_{W_2^{0,r/2}(Q_T)},$$

where

$$\|u\|^2_{W_2^{r,0}(Q_T)} \overset{\text{def}}{=} \int_0^T \|u\|^2_{W_2^r(\Omega)}\, dt, \qquad \|u\|^2_{W_2^{0,r/2}(Q_T)} \overset{\text{def}}{=} \int_\Omega \|u\|^2_{W_2^{r/2}(0,T)}\, dx.$$

Notice that the spaces $W_2^r(\Gamma)$ and $W_2^{r,r/2}(G_T)$ (here $G_T \overset{\text{def}}{=} \Gamma \times (0,T)$) can be defined in the usual way by means of local charts and a partition of unity on Γ.

Besides the function spaces $W_2^r(\Omega), W_2^{r,r/2}(Q_T)$, given $\gamma > 0$, the space $H_\gamma^{r,r/2}(Q_T)$ will be very useful in what follows, which is defined as the set of functions on Q_T with finite norm

$$\|u\|^2_{H_\gamma^{r,r/2}(Q_T)} \overset{\text{def}}{=} \|u\|^2_{H_\gamma^{r,0}(Q_T)} + \|u\|^2_{H_\gamma^{0,r/2}(Q_T)},$$

where

$$\|u\|^2_{H^{r,0}_\gamma(Q_T)} \stackrel{\text{def}}{=} \int_0^T e^{-2\gamma t}\|u\|^2_{\dot W^r_2(\Omega)}\,dt,$$

and

$$\|u\|^2_{H^{0,r/2}_\gamma(Q_T)} \stackrel{\text{def}}{=} \begin{cases} \gamma^r \int_0^T e^{-2\gamma t}\|u\|^2_{L^2(\Omega)}\,dt \\ \quad + \int_0^T e^{-2\gamma t}\,dt \int_0^\infty \| \frac{\partial^k u_0(\cdot,t-\tau)}{\partial t^k} - \frac{\partial^k u_0(\cdot,t)}{\partial t^k}\|^2_{L^2(\Omega)} \frac{d\tau}{\tau^{1+r-2k}} & \text{if } k=[r/2]<r/2, \\ \int_0^T e^{-2\gamma t}(\gamma^r\|u\|^2_{L^2(\Omega)} + \|\frac{\partial^{r/2}u}{\partial t^{r/2}}\|^2_{L^2(\Omega)})\,dt & \text{if } [r/2]=r/2. \end{cases}$$

Here $u_0(x,t)=u(x,t)$ for $t>0$, $u_0(x,t)=0$ for $t<0$, and we assume that

$$\begin{aligned} &\frac{\partial^j u}{\partial t^j}\bigg|_{t=0} = 0, \quad j=0,\dots,\left[\frac{r-1}{2}\right], \quad \text{if } k=\left[\frac{r}{2}\right]<\frac{r}{2},\ r \text{ is not odd}, \\ &\frac{\partial^j u}{\partial t^j}\bigg|_{t=0} = 0, \quad j=0,\dots,\frac{r-1}{2}-1, \quad \text{if } k=\left[\frac{r}{2}\right]<\frac{r}{2},\ r \text{ odd}, \\ &\frac{\partial^j u}{\partial t^j}\bigg|_{t=0} = 0, \quad j=0,\dots,\frac{r}{2}-1, \quad \text{otherwise}, \end{aligned}$$

which ensure that $u(x,t)$ can be extended by zero into the domain $t<0$ without loss of smoothness.

We further evoke the norm of $H^{r+1/2,1/2,r/2}_\gamma(G_T)$ by

$$\begin{aligned} \|u\|^2_{H^{r+1/2,1/2,r/2}_\gamma(G_T)} \stackrel{\text{def}}{=} {}& \int_0^T e^{-2\gamma t}\big(\|u\|^2_{W^{r+1/2}_2(\Gamma)} + \gamma^r\|u\|^2_{W^{1/2}_2(\Gamma)}\big)\,dt \\ & + \int_0^T e^{-2\gamma t}\,dt \int_0^\infty \left\| \frac{\partial^k u_0(\cdot,t-\tau)}{\partial t^k} - \frac{\partial^k u_0(\cdot,t)}{\partial t^k}\right\|^2_{W^{1/2}_2(\Gamma)} \frac{d\tau}{\tau^{1+r-2k}}, \end{aligned}$$

if $k=[r/2]<r/2$, and $\frac{\partial^j u}{\partial t^j}|_{t=0}=0$, $j=0,\dots,[\frac{r-1}{2}]$ if r is not an odd integer, while we assume that $\frac{\partial^j u}{\partial t^j}|_{t=0}=0$, $j=0,\dots,\frac{r-1}{2}-1$ if r is an odd integer. On the other hand, when $r/2$ is an integer, and $\frac{\partial^j u}{\partial t^j}|_{t=0}=0$, $j=0,\dots,r/2-1$, we define its norm as

$$\|u\|^2_{H^{r+1/2,1/2,r/2}_\gamma(G_T)} \stackrel{\text{def}}{=} \int_0^T e^{-2\gamma t}\left(\|u\|^2_{W^{r+1/2}_2(\Gamma)} + \gamma^r\|u\|^2_{W^{1/2}_2(\Gamma)} + \left\|\frac{\partial^{r/2}u}{\partial t^{r/2}}\right\|^2_{W^{1/2}_2(\Gamma)}\right)dt.$$

In particular, when $\Gamma=\mathbb{R}^{n-1}$, $\mathbb{R}^{n-1}_T \stackrel{\text{def}}{=} \mathbb{R}^{n-1}\times(0,T)$, we set

$$\|u\|^2_{H^{r+1/2,1/2,r/2}_\gamma(\mathbb{R}^{n-1}_T)} \stackrel{\text{def}}{=} \int_0^T e^{-2\gamma t}\big(\|u\|^2_{\dot W^{r+1/2}_2(\mathbb{R}^{n-1})} + \gamma^r\|u\|^2_{\dot W^{1/2}_2(\mathbb{R}^{n-1})}\big)\,dt$$

$$+\int_0^T e^{-2\gamma t}\,dt\int_0^\infty \left\| \frac{\partial^k u_0(\cdot,t-\tau)}{\partial t^k}-\frac{\partial^k u_0(\cdot,t)}{\partial t^k}\right\|^2_{\dot W_2^{1/2}(\mathbb{R}^{n-1})}\frac{d\tau}{\tau^{1+r-2k}},$$

if $r/2$ is not an integer, and

$$\|u\|^2_{H_\gamma^{r+1/2,1/2,r/2}(\mathbb{R}_T^{n-1})}\overset{\text{def}}{=}\int_0^T e^{-2\gamma t}\left(\|u\|^2_{\dot W_2^{r+1/2}(\mathbb{R}^{n-1})}+\gamma^r\|u\|^2_{\dot W_2^{1/2}(\mathbb{R}^{n-1})}+\left\|\frac{\partial^{r/2}u}{\partial t^{r/2}}\right\|^2_{\dot W_2^{1/2}(\mathbb{R}^{n-1})}\right)dt,$$

if $r/2$ is an integer.

Next, we list some well-known properties of the function spaces $H_\gamma^{r,r/2}(Q_T)$ for the bounded domain $\Omega\subset\mathbb{R}^3$ from [1,15].

Lemma 2.1. *(See [15, Lemma 4.1].) Let $l>1/2$, $a\in W_2^{l+1}(\Omega), b\in W_2^l(\Omega), f\in H_\gamma^{l,l/2}(Q_T)$ and $g\in H_\gamma^{l+1,l/2+1/2}(Q_T)$. Then there hold*

$$\|af\|_{H_\gamma^{l,l/2}(Q_T)}\leqslant\|f\|_{H_\gamma^{l,l/2}(Q_T)}\Big(C\sup_\Omega|a|+\big(\epsilon+C(\epsilon)\gamma^{-l/2}\big)\|a\|_{W_2^{l+1}(\Omega)}\Big),$$

$$\|bg\|_{H_\gamma^{l,l/2}(Q_T)}\leqslant C\|b\|_{W_2^l(\Omega)}\big(\epsilon+C(\epsilon)\gamma^{-l/2}\big)\|g\|_{H_\gamma^{l+1,l/2+1/2}(Q_T)},$$

$$\|ag\|_{H_\gamma^{l+1,l/2+1/2}(Q_T)}\leqslant\|g\|_{H_\gamma^{l+1,l/2+1/2}(Q_T)}\Big(C\sup_\Omega|a|+\big(\epsilon+C(\epsilon)\gamma^{-(l+1)/2}\big)\|a\|_{W_2^{l+1}(\Omega)}\Big).$$

The following inequalities from [15] will be constantly used in this paper:

$$\big\|D_t^m D_x^\alpha u\big\|_{H_\gamma^{r_1,r_1/2}(Q_T)}\leqslant C\|u\|_{H_\gamma^{r,r/2}(Q_T)},\tag{2.1}$$

$$\big\|D_t^m D_x^\alpha u\big\|_{H_\gamma^{\rho,\rho/2}(Q_T)}\leqslant\big(\epsilon+C\gamma^{-\frac{r}{2}}\epsilon^{-\frac{\rho+2m+|\alpha|}{r_1-\rho}}\big)\|u\|_{H_\gamma^{r,r/2}(Q_T)},\tag{2.2}$$

for $r_1=r-2m-|\alpha|\in(0,r)$, $\rho\in(0,r_1)$, while

$$\big\|D_t^m D_x^\alpha u\big|_\Gamma\big\|_{H_\gamma^{r_1-1/2,r_1/2-1/4}(G_T)}\leqslant C\|u\|_{H_\gamma^{r,r/2}(Q_T)}\tag{2.3}$$

for $r_1\in(1/2,r)$, and

$$\|u\|_{H_\gamma^{r-1/2,1/2,r/2-1/2}(G_T)}\leqslant C\|u\|_{H_\gamma^{r-1/2,r/2-1/4}(G_T)}.\tag{2.4}$$

3. The solvability of linearized system

One crucial step in the proof of the main results is to study the solvability of the following linearized problem of (1.8)–(1.9):

$$\begin{cases}\rho_0^{(j)}u_t^{(j)}-\nu^{(j)}\nabla^2u^{(j)}+\nabla q^{(j)}=\rho_0^{(j)}f^{(j)},\quad \nabla\cdot u^{(j)}=g^{(j)}\quad\text{in }Q_T^{(j)},\\ u^{(j)}\big|_{t=0}=v_0^{(j)}\quad\text{in }\Omega^{(j)},\ j=1,2,\end{cases}\tag{3.1}$$

together with the boundary conditions:

$$\begin{cases} \Pi^{(1)}\big[\nu^{(1)}\mathrm{D}\big(u^{(1)}\big)N^{(1)}\big]\big|_{S^{(1)}_{F,T}} = \vec{b}^{(1)}, \\ -q^{(1)} + 2\nu^{(1)}N^{(1)}\cdot\big(\nabla u^{(1)}N^{(1)}\big) - \sigma^{(1)}N^{(1)}\cdot\Delta^{(1)}\displaystyle\int_0^t u^{(1)}\,ds\Big|_{S^{(1)}_{F,T}} \\ \quad = \bar{b}^{(1)} + \sigma^{(1)}\displaystyle\int_0^t B^{(1)}\,ds, \\ \Pi^{(2)}\big[\nu^{(2)}\mathrm{D}\big(u^{(2)}\big)N^{(2)} - \nu^{(1)}\mathrm{D}\big(u^{(1)}\big)N^{(2)}\big]\big|_{S^{(2)}_{F,T}} = \vec{b}^{(2)}, \\ q^{(1)} - q^{(2)} + 2\nu^{(2)}N^{(2)}\cdot\big(\nabla u^{(2)}N^{(2)}\big) - 2\nu^{(1)}N^{(2)}\cdot\big(\nabla u^{(1)}N^{(2)}\big) \\ \quad -\dfrac{\sigma^{(2)}}{2}N^{(2)}\cdot\Delta^{(2)}\displaystyle\int_0^t\big(u^{(1)}+u^{(2)}\big)\,ds\Big|_{S^{(2)}_{F,T}} = \bar{b}^{(2)} + \dfrac{\sigma^{(2)}}{2}\displaystyle\int_0^t B^{(2)}\,ds, \\ u^{(1)} - u^{(2)}\big|_{S^{(2)}_{F,T}} = 0, \quad u^{(2)}\big|_{S_{B,T}} = 0, \end{cases} \tag{3.2}$$

where $Q_T^{(j)} = \Omega^{(j)} \times (0,T)$, $S^{(j)}_{F,T} = S^{(j)}_F \times (0,T)$, $\Pi^{(j)}f = f - (f\cdot N^{(j)})N^{(j)}$ and $\Delta^{(j)} = \Delta^{(j)}(0)$ is the Laplace–Beltrami operator on $S^{(j)}_F$.

Theorem 3.1. *Let $l \in (\frac{1}{2}, 1)$, $\sigma^{(j)} > 0$, $\nu^{(j)} > 0$. Given $T > 0$, $S^{(1)}_F, S^{(2)}_F, S_B \in W_2^{l+\frac{3}{2}}$, $\nabla\rho^{(j)}_0 \in W^l_2(\Omega^{(j)})$ with*

$$M^{(j)}_1 \leqslant \rho^{(j)}_0 \leqslant M^{(j)}_2,$$

for some positive constants M^j_i, $i, j = 1, 2$, $v^{(j)}_0 \in W^{l+1}_2(\Omega^{(j)})$,

$$\begin{aligned} \big(f^{(j)}, g^{(j)}, \vec{b}^{(j)}, \bar{b}^{(j)}, B^{(j)}\big) \in{}& W_2^{l,l/2}\big(Q^{(j)}_T\big) \times W_2^{l+1,(l+1)/2}\big(Q^{(j)}_T\big) \times W_2^{l+1/2,l/2+1/4}\big(S^{(j)}_{F,T}\big) \\ &\times W_2^{l+1/2,l/2+1/4}\big(S^{(j)}_{F,T}\big) \times W_2^{l-1/2,l/2-1/4}\big(S^{(j)}_{F,T}\big) \end{aligned}$$

and $g^{(j)} = \nabla\cdot R^{(j)}$ with $R^{(j)} \in L^2(Q^{(j)}_T)$, $R^{(j)}_t \in W_2^{0,l/2}(Q^{(j)}_T)$. Moreover, we assume that the following compatibility conditions are satisfied:

$$\vec{b}^{(j)}\cdot N^{(j)} = 0, \qquad \nabla\cdot v^{(j)}_0 = g^{(j)}\big|_{t=0}, \qquad v^{(1)}_0 - v^{(2)}_0\big|_{S^{(2)}_F} = 0, \qquad v^{(2)}_0\big|_{S_B} = 0,$$
$$\vec{b}^{(1)}\big|_{t=0} = \Pi^{(1)}\big[\nu^{(1)}\mathrm{D}\big(v^{(1)}_0\big)N^{(1)}\big]\big|_{S^{(1)}_F},$$
$$\vec{b}^{(2)}\big|_{t=0} = \Pi^{(2)}\big[\nu^{(2)}\mathrm{D}\big(v^{(2)}_0\big)N^{(2)} - \nu^{(1)}\mathrm{D}\big(v^{(1)}_0\big)N^{(2)}\big]\big|_{S^{(2)}_F}.$$

Then the problem (3.1)–(3.2) *has a unique solution $(u^{(1)}, q^{(1)}, u^{(2)}, q^{(2)})$ with $(u^{(j)}, q^{(j)}, \nabla q^{(j)}) \in W_2^{l+2,l/2+1}(Q^{(j)}_T) \times W_2^{l,l/2}(Q^{(j)}_T) \times W_2^{l,l/2}(Q^{(j)}_T)$, $q^{(1)}|_{S^{(1)}_{F,T}} \in W_2^{l+1/2,l/2+1/4}(S^{(1)}_{F,T})$, $q^{(1)} - q^{(2)}|_{S^{(2)}_{F,T}} \in W_2^{l+1/2,l/2+1/4}(S^{(2)}_{F,T})$, and there holds*

$$\sum_{j=1}^{2}\big(\|u^{(j)}\|^{(l+2)}_{Q^{(j)}_T} + \|\nabla q^{(j)}\|^{(l)}_{Q^{(j)}_T} + \|q^{(j)}\|^{(l)}_{Q^{(j)}_T}\big)$$

$$
\begin{aligned}
&+ \|q^{(1)}\|_{W_2^{l+1/2,l/2+1/4}(S_{F,T}^{(1)})} + \|q^{(1)} - q^{(2)}\|_{W_2^{l+1/2,l/2+1/4}(S_{F,T}^{(2)})} \\
\leqslant\ & C(T) \sum_{j=1}^{2} \Big[\|f^{(j)}\|_{Q_T^{(j)}}^{(l)} + \|g^{(j)}\|_{W_2^{l+1,l/2+1/2}(Q_T^{(j)})} + \|R^{(j)}\|_{W_2^{0,l/2+1}(Q_T^{(j)})} \\
&+ T^{-l/2}\|R^{(j)}\|_{L^2(Q_T^{(j)})} + \big\|\big(\vec{b}^{(j)}, \bar{b}^{(j)}\big)\big\|_{W_2^{l+1/2,l/2+1/4}(S_{F,T}^{(j)})} \\
&+ T^{-l/2}\|\bar{b}^{(j)}\|_{W_2^{1/2,0}(S_{F,T}^{(j)})} + \sigma^{(j)}\|B^{(j)}\|_{S_{F,T}^{(j)}}^{(l-1/2)} + \|v_0^{(j)}\|_{W_2^{l+1}(\Omega^{(j)})}\Big] \\
\stackrel{\text{def}}{=}\ & C(T)\Xi, \qquad (3.3)
\end{aligned}
$$

where $C(T)$ is an increasing function of T, and

$$
\begin{aligned}
\|u\|_{Q_T}^{(l+2)\,2} &\stackrel{\text{def}}{=} \|u_t\|_{Q_T}^{(l)\,2} + \sum_{|s|=2} \|D_x^s u\|_{Q_T}^{(l)\,2} + \sum_{|s|=0}^{1} \|D_x^s u\|_{L^2(Q_T)}^2, \\
\|u\|_{Q_T}^{(l)\,2} &\stackrel{\text{def}}{=} \|u\|_{W_2^{l,l/2}(Q_T)}^2 + T^{-l}\|u\|_{L^2(Q_T)}^2, \\
\|B\|_{S_{F,T}}^{(r)\,2} &\stackrel{\text{def}}{=} \|B\|_{W_2^{r,r/2}(S_{F,T})}^2 + T^{-r}\|B\|_{L^2(S_{F,T})}^2 \quad (0 < r < 1).
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
&\sum_{j=1}^{2} \big(\|u^{(j)}\|_{W_2^{l+2,l/2+1}(Q_T^{(j)})} + \|\nabla q^{(j)}\|_{W_2^{l,l/2}(Q_T^{(j)})} + \|q^{(j)}\|_{W_2^{l,l/2}(Q_T^{(j)})} \big) \\
&\quad + \|q^{(1)}\|_{W_2^{l+1/2,l/2+1/4}(S_{F,T}^{(1)})} + \|q^{(1)} - q^{(2)}\|_{W_2^{l+1/2,l/2+1/4}(S_{F,T}^{(2)})} \\
&\leqslant C(T) \sum_{j=1}^{2} \Big[\|f^{(j)}\|_{W_2^{l,l/2}(Q_T^{(j)})} + \|g^{(j)}\|_{W_2^{l+1,l/2+1/2}(Q_T^{(j)})} \\
&\quad + \|R^{(j)}\|_{W_2^{0,l/2+1}(Q_T^{(j)})} + \big\|\big(\vec{b}^{(j)}, \bar{b}^{(j)}\big)\big\|_{W_2^{l+1/2,l/2+1/4}(S_{F,T}^{(j)})} \\
&\quad + \sigma^{(j)}\|B^{(j)}\|_{W_2^{l-1/2,l/2-1/4}(S_{F,T}^{(j)})} + \|v_0^{(j)}\|_{W_2^{l+1}(\Omega^{(j)})}\Big] \\
&\stackrel{\text{def}}{=} C(T)\Xi'. \qquad (3.4)
\end{aligned}
$$

3.1. The presentation of the auxiliary problems

We shall first seek the solution to (3.1)–(3.2) with $v_0^{(j)} = 0$ in the form $(u^{(j)}, q^{(j)}) = (u'^{(j)} + u''^{(j)}, q'^{(j)} + q''^{(j)})$, where $(u'^{(j)}, q'^{(j)})$ satisfies

$$
\begin{cases}
\rho_0^{(j)} u_t'^{(j)} - \nu^{(j)} \nabla^2 u'^{(j)} + \nabla q'^{(j)} = \rho_0^{(j)} f^{(j)}, \quad \nabla \cdot u'^{(j)} = g^{(j)} \quad \text{in } Q_T^{(j)}, \\
u'^{(j)}\big|_{t=0} = 0 \quad \text{in } \Omega^{(j)},\ j = 1, 2,
\end{cases} \qquad (3.5)
$$

together with the boundary conditions

$$\begin{cases} \Pi^{(1)}\big[\nu^{(1)}\mathrm{D}\big(u'^{(1)}\big)N^{(1)}\big]\big|_{S^{(1)}_{F,T}} = \vec{b}^{(1)}, \\ -q'^{(1)} + 2\nu^{(1)}N^{(1)}\cdot\big(\nabla u'^{(1)}N^{(1)}\big)\big|_{S^{(1)}_{F,T}} = \bar{b}^{(1)}, \\ \Pi^{(2)}\big[\nu^{(2)}\mathrm{D}\big(u'^{(2)}\big)N^{(2)} - \nu^{(1)}\mathrm{D}\big(u'^{(1)}\big)N^{(2)}\big]\big|_{S^{(2)}_{F,T}} = \vec{b}^{(2)}, \\ q'^{(1)} - q'^{(2)} + 2\nu^{(2)}N^{(2)}\cdot\big(\nabla u'^{(2)}N^{(2)}\big) - 2\nu^{(1)}N^{(2)}\cdot\big(\nabla u'^{(1)}N^{(2)}\big)\big|_{S^{(2)}_{F,T}} = \bar{b}^{(2)}, \\ u'^{(1)} - u'^{(2)}\big|_{S^{(2)}_{F,T}} = 0, \qquad u'^{(2)}\big|_{S_{B,T}} = 0, \end{cases} \tag{3.6}$$

and $(u''^{(j)}, q''^{(j)})$ satisfies

$$\begin{cases} \rho_0^{(j)}u_t''^{(j)} - \nu^{(j)}\nabla^2 u''^{(j)} + \nabla q''^{(j)} = 0, \quad \nabla\cdot u''^{(j)} = 0 \quad \text{in } Q_T^{(j)}, \\ u''^{(j)}\big|_{t=0} = 0 \quad \text{in } \Omega^{(j)},\ j = 1, 2, \end{cases} \tag{3.7}$$

together with the boundary conditions

$$\begin{cases} \Pi^{(1)}\big[\nu^{(1)}\mathrm{D}\big(u''^{(1)}\big)N^{(1)}\big]\big|_{S^{(1)}_{F,T}} = 0, \\ -q''^{(1)} + 2\nu^{(1)}N^{(1)}\cdot\big(\nabla u''^{(1)}N^{(1)}\big) - \sigma^{(1)}N^{(1)}\cdot\Delta^{(1)}\displaystyle\int_0^t u''^{(1)}\,ds\Big|_{S^{(1)}_{F,T}} \\ \quad = \sigma^{(1)}\displaystyle\int_0^t \big(B^{(1)} + N^{(1)}\cdot\Delta^{(1)}u'^{(1)}\big)\,ds \overset{\mathrm{def}}{=} \sigma^{(1)}\displaystyle\int_0^t D^{(1)}\,ds, \\ \Pi^{(2)}\big[\nu^{(2)}\mathrm{D}\big(u''^{(2)}\big)N^{(2)} - \nu^{(1)}\mathrm{D}\big(u''^{(1)}\big)N^{(2)}\big]\big|_{S^{(2)}_{F,T}} = 0, \\ q''^{(1)} - q''^{(2)} + 2\nu^{(2)}N^{(2)}\cdot\big(\nabla u''^{(2)}N^{(2)}\big) - 2\nu^{(1)}N^{(2)}\cdot\big(\nabla u''^{(1)}N^{(2)}\big) \\ \quad - \dfrac{\sigma^{(2)}}{2}N^{(2)}\cdot\Delta^{(2)}\displaystyle\int_0^t\big(u''^{(1)} + u''^{(2)}\big)\,ds\Big|_{S^{(2)}_{F,T}}, \\ \quad = \dfrac{\sigma^{(2)}}{2}\displaystyle\int_0^t\big(B^{(2)} + N^{(2)}\cdot\Delta^{(2)}\big(u'^{(1)} + u'^{(2)}\big)\big)\,ds \overset{\mathrm{def}}{=} \dfrac{\sigma^{(2)}}{2}\displaystyle\int_0^t D^{(2)}\,ds, \\ u''^{(1)} - u''^{(2)}\big|_{S^{(2)}_{F,T}} = 0, \quad u''^{(2)}\big|_{S_{B,T}} = 0. \end{cases} \tag{3.8}$$

Then the main existence results concerning (3.5)–(3.6) and (3.7)–(3.8) can be listed as follows:

Theorem 3.2. *Let $l > \frac{1}{2}$, $T > 0$, $\gamma \geqslant 0$ and $\nu^{(j)} > 0$. Let $S_F^{(1)}$, $S_F^{(2)}$, $S_B \in W_2^{l+3/2}$, and $\nabla\rho_0^{(j)} \in W_2^l(\Omega^{(j)})$ with $\rho_0^{(j)}$ satisfying*

$$M_1^{(j)} \leqslant \rho_0^{(j)} \leqslant M_2^{(j)},$$

for some positive constants $M_i^{(j)}$, $i, j = 1, 2$. Then for any

$$\big(f^{(j)}, g^{(j)}, \vec{b}^{(j)}, \bar{b}^{(j)}\big) \in H_\gamma^{l,l/2}\big(Q_T^{(j)}\big) \times H_\gamma^{l+1,(l+1)/2}\big(Q_T^{(j)}\big) \\ \times H_\gamma^{l+1/2,l/2+1/4}\big(S_{F,T}^{(j)}\big) \times H_\gamma^{l+1/2,1/2,l/2}\big(S_{F,T}^{(j)}\big)$$

satisfying the compatibility conditions $\vec{b}^{(j)} \cdot N^{(j)} = 0$, $(g^{(j)}, \vec{b}^{(j)})|_{t=0} = 0$ *and* $g^{(j)} = \nabla \cdot R^{(j)}$ *with* $R^{(j)} \in H_\gamma^{0,l/2+1}(Q_T^{(j)})$, (3.5)–(3.6) *has a unique solution* $(u'^{(1)}, q'^{(1)}, u'^{(2)}, q'^{(2)})$ *with* $(u'^{(j)}, q'^{(j)}) \in H_\gamma^{l+2,l/2+1}(Q_T^{(j)}) \times H_\gamma^{l+1,1,l/2}(Q_T^{(j)})$ *for* $\gamma \geqslant \gamma_0 \gg 1$. *Moreover, there holds*

$$\begin{aligned}
&\sum_{j=1}^{2}\bigl(\|u'^{(j)}\|_{H_\gamma^{l+2,l/2+1}(Q_T^{(j)})} + \|q'^{(j)}\|_{H_\gamma^{l+1,1,l/2}(Q_T^{(j)})}\bigr) \\
&\quad\leqslant C\sum_{j=1}^{2}\bigl[\|f^{(j)}\|_{H_\gamma^{l,l/2}(Q_T^{(j)})} + \|g^{(j)}\|_{H_\gamma^{l+1,0}(Q_T^{(j)})} + \|R^{(j)}\|_{H_\gamma^{0,l/2+1}(Q_T^{(j)})} \\
&\qquad + \|\vec{b}^{(j)}\|_{H_\gamma^{l+1/2,l/2+1/4}(S_{F,T}^{(j)})} + \|\bar{b}^{(j)}\|_{H_\gamma^{l+1/2,1/2,l/2}(S_{F,T}^{(j)})}\bigr] \stackrel{\text{def}}{=} C\Xi_1,
\end{aligned} \tag{3.9}$$

where

$$\|f\|^2_{H_\gamma^{l+1,1,l/2}(Q_T)} \stackrel{\text{def}}{=} \|f\|^2_{H_\gamma^{l,l/2}(Q_T)} + \|\nabla f\|^2_{H_\gamma^{l,l/2}(Q_T)}.$$

Theorem 3.3. *Under the same assumptions in Theorem* 3.2, *and we assume further that* $\sigma^{(j)} > 0$. *Then given any*

$$D^{(j)} \in H_\gamma^{l-1/2,l/2-1/4}\bigl(S_{F,T}^{(j)}\bigr),$$

(3.7)–(3.8) *has a unique solution* $(u''^{(1)}, q''^{(1)}, u''^{(2)}, q''^{(2)})$ *such that* $(u''^{(j)}, q''^{(j)}) \in H_\gamma^{l+2,l/2+1}(Q_T^{(j)}) \times H_\gamma^{l+1,1,l/2}(Q_T^{(j)})$, $j = 1, 2$, *for* $\gamma \geqslant \gamma_0 \gg 1$, *and there holds*

$$\sum_{j=1}^{2}\bigl(\|u''^{(j)}\|_{H_\gamma^{l+2,l/2+1}(Q_T^{(j)})} + \|q''^{(j)}\|_{H_\gamma^{l+1,1,l/2}(Q_T^{(j)})}\bigr) \leqslant C\sum_{j=1}^{2}\sigma^{(j)}\|D^{(j)}\|_{H_\gamma^{l-1/2,l/2-1/4}(S_{F,T}^{(j)})}. \tag{3.10}$$

We shall first prove Theorem 3.3 in following subsection by applying Theorem 3.2. And in the last subsection, we present the proof of Theorem 3.2.

3.2. Proof of Theorem 3.3

The main ideas to the proof of Theorem 3.3 come from [15,16] and [19,21]. We shall first construct a continuous linear operator

$$\begin{aligned}
&\mathscr{L} : H_\gamma^{l-1/2,l/2-1/4}\bigl(S_{F,T}^{(1)}\bigr) \times H_\gamma^{l-1/2,l/2-1/4}\bigl(S_{F,T}^{(2)}\bigr) \\
&\quad \to H_\gamma^{l+2,l/2+1}\bigl(Q_T^{(1)}\bigr) \times H_\gamma^{l+1,1,l/2}\bigl(Q_T^{(1)}\bigr) \times H_\gamma^{l+2,l/2+1}\bigl(Q_T^{(2)}\bigr) \times H_\gamma^{l+1,1,l/2}\bigl(Q_T^{(2)}\bigr)
\end{aligned}$$

such that $(u''^{(1)}, q''^{(1)}, u''^{(2)}, q''^{(2)}) = \mathscr{L}D$ satisfies (3.7)–(3.8) with $(D^{(1)}, D^{(2)})$ there being replaced by $(D^{(1)} + \mathscr{M}_1 D, D^{(2)} + \mathscr{M}_2 D)$ where $D = (D^{(1)}, D^{(2)})$, and $\mathscr{M} = (\mathscr{M}_1, \mathscr{M}_2)$ is a continuous linear operator on $H_\gamma^{l-1/2,l/2-1/4}(S_{F,T}^{(1)}) \times H_\gamma^{l-1/2,l/2-1/4}(S_{F,T}^{(2)})$. Then we shall show that $\mathscr{M}$ becomes a contraction if $\gamma \gg 1$, which implies the solvability of (3.7)–(3.8) and the solution $(u''^{(1)}, q''^{(1)}, u''^{(2)}, q''^{(2)}) = \mathscr{L}(I + \mathscr{M})^{-1}D$.

In order to do so, it is necessary to introduce the covering systems of $\bar{\Omega} \stackrel{\text{def}}{=} \bar{\Omega}^{(1)} \cup \bar{\Omega}^{(2)}$ as follows (see [19,21]): firstly the assumption that $S_F^{(0)} \stackrel{\text{def}}{=} S_B$, $S_F^{(1)}, S_F^{(2)} \in W_2^{l+3/2}$, implies that in the neighborhood of any $\xi \in S_F^{(j)}$, $S_F^{(j)}$ can be reexpressed as

$$y_3 = \varphi^{(j)}\big(y'\big), \quad y' = (y_1, y_2) \in K_{r^{(j)}}, \ \varphi^{(j)} \in W_2^{l+3/2}(K_{r^{(j)}}), \ j = 0, 1, 2,$$

in a Cartesian coordinate system $\{y\} = \{y', y_3\}$ with the original at ξ and with y_3-axis being directed along $-N^{(1)}$ for $S_F^{(1)}$, and with y_3-axis being directed along $N^{(2)}$ and $N^{(0)}$ for $S_F^{(2)}$ and $S_F^{(0)}$ respectively. Here $N^{(0)} = N_B$ is a unit inward normal to S_B. And the functions $\varphi^{(j)}$, $j = 0, 1, 2$, are defined in $K_{r^{(j)}} = \{y' \in \mathbb{R}^2 \mid |y'| < r^{(j)}\}$, which satisfy

$$\varphi^{(j)}(0) = 0, \qquad \nabla_{y'}\varphi^{(j)}(0) = 0, \qquad \big\|\varphi^{(j)}\big\|_{W_2^{l+3/2}(K_{r^{(j)}})} \leqslant M^{(j)}. \tag{3.11}$$

Here and in what follows, we assume that $r^{(j)}, M^{(j)}$ are independent of ξ, and we denote $r \stackrel{\text{def}}{=} \min\{r^{(j)};\ j = 0, 1, 2\}$ and $M \stackrel{\text{def}}{=} \max\{M^{(j)};\ j = 0, 1, 2\}$.

For any $0 < \lambda < r$, we introduce two covering systems $\{\omega_k\}$ and $\{\Omega_k\}$ of $\bar{\Omega}$ as follows:

1. $\omega_k \subset \Omega_k \subset \bar{\Omega}$, $\bigcup_k \omega_k = \bigcup_k \Omega_k = \bar{\Omega}$.
2. $\forall \xi \in \bar{\Omega}$, $\exists \omega_k$, s.t. $\xi \in \omega_k$ and $\operatorname{dist}(\xi, \bar{\Omega} \backslash \omega_k) \geqslant \beta_1 \lambda$ for some $\beta_1 > 0$.
3. $\forall \lambda > 0$, $\exists N_0 \in \mathbb{N}$ independent of λ, s.t. $\bigcap_{k=1}^{N_0+1} \Omega_k = \emptyset$.

4(i). If $\Omega_k \cap (S_F^{(1)} \cup S_F^{(2)}) = \emptyset$ or $\Omega_k \cap (S_F^{(2)} \cup S_F^{(0)}) = \emptyset$ (say $k = k'$), then ω_k and Ω_k are cubes with the same center $\xi^{(k')}$ and with the length of their edges equal to $\lambda/2$ and λ, respectively.

4(ii)$^{(1)}$. If $\omega_k \cap S_F^{(1)} \neq \emptyset$ and $\Omega_k \cap S_F^{(2)} = \emptyset$ (say $k = k''$), then in the local rectangular coordinate system $\{y\}$ with the original at $\xi^{(k'')} \in S_F^{(1)}$, $(\omega_{k''}, \Omega_{k''}) = Y_{k''}^{-1}(\hat{\omega}_{k''}, \hat{\Omega}_{k''})$ where $y = Y_{k''}(x) = L_{k''}(x - \xi^{(k'')})$, $L_{k''}$ is an orthogonal matrix and

$$\hat{\omega}_{k''} = \big\{y \in \mathbb{R}^3 \,\big|\, |y_j| \leqslant \beta_2\lambda/2,\ j = 1, 2,\ 0 \leqslant y_3 - \varphi^{(1)}\big(y'; \xi^{(k'')}\big) \leqslant \beta_2\lambda\big\},$$
$$\hat{\Omega}_{k''} = \big\{y \in \mathbb{R}^3 \,\big|\, |y_j| \leqslant \beta_2\lambda,\ j = 1, 2,\ 0 \leqslant y_3 - \varphi^{(1)}\big(y'; \xi^{(k'')}\big) \leqslant 2\beta_2\lambda\big\}.$$

Here $y_3 = \varphi^{(1)}(y'; \xi^{(k'')})$ represents the $S_F^{(1)}$ in the neighborhood of $\xi^{(k'')}$, and $\beta_2 > 0$ is independent of λ and $\xi^{(k'')}$.

4(ii)$^{(2)}$. If $\omega_k \cap S_F^{(2)} \neq \emptyset$, $\Omega_k \cap S_F^{(0)} = \emptyset$ and $\Omega_k \cap S_F^{(1)} = \emptyset$ (say $k = k'''$), then $(\omega_{k'''}, \Omega_{k'''}) = (\bigcup_{i=1}^2 \omega_{k'''}^{(i)}, \bigcup_{i=1}^2 \Omega_{k'''}^{(i)}) = \bigcup_{i=1}^2 Y_{k'''}^{-1}(\hat{\omega}_{k'''}^{(i)}, \hat{\Omega}_{k'''}^{(i)})$ where

$$\hat{\omega}_{k'''}^{(1)} = \big\{y \in \mathbb{R}^3 \,\big|\, |y_j| \leqslant \beta_2\lambda/2,\ j = 1, 2,\ 0 \leqslant y_3 - \varphi^{(2)}\big(y'; \xi^{(k''')}\big) \leqslant \beta_2\lambda\big\},$$
$$\hat{\Omega}_{k'''}^{(1)} = \big\{y \in \mathbb{R}^3 \,\big|\, |y_j| \leqslant \beta_2\lambda,\ j = 1, 2,\ 0 \leqslant y_3 - \varphi^{(2)}\big(y'; \xi^{(k''')}\big) \leqslant 2\beta_2\lambda\big\},$$
$$\hat{\omega}_{k'''}^{(2)} = \big\{y \in \mathbb{R}^3 \,\big|\, |y_j| \leqslant \beta_2\lambda/2,\ j = 1, 2,\ 0 \leqslant \varphi^{(2)}\big(y'; \xi^{(k''')}\big) - y_3 \leqslant \beta_2\lambda\big\},$$
$$\hat{\Omega}_{k'''}^{(2)} = \big\{y \in \mathbb{R}^3 \,\big|\, |y_j| \leqslant \beta_2\lambda,\ j = 1, 2,\ 0 \leqslant \varphi^{(2)}\big(y'; \xi^{(k''')}\big) - y_3 \leqslant 2\beta_2\lambda\big\},$$

and $y_3 = \varphi^{(2)}(y'; \xi^{(k''')})$ represents the $S_F^{(2)}$ in the neighborhood of $\xi^{(k''')}$.

4(ii)$^{(0)}$. If $\omega_k \cap S_F^{(0)} \neq \emptyset$ and $\Omega_k \cap S_F^{(2)} = \emptyset$ (say $k = k^{(4)}$), then $\omega_{k^{(4)}}$ and $\Omega_{k^{(4)}}$ can be determined in the same way as in 4(ii)$^{(1)}$ with $S_F^{(1)}$ and $\varphi^{(1)}$ there being replaced by $S_F^{(0)}$ and $\varphi^{(0)}$ respectively.

We now choose λ small enough such that $2\beta_2\lambda \leqslant r$. Let $\{\zeta_k(x)\}$ be smooth functions satisfying

$$\zeta_k(x) = \begin{cases} 1, & x \in \omega_k, \\ 0, & x \in \bar{\Omega}\backslash\Omega_k, \end{cases} \quad \text{and}$$
$$\left|D^\alpha \zeta_k(x)\right| \leqslant C_\alpha \lambda^{-|\alpha|}, \quad 0 \leqslant \zeta_k(x) \leqslant 1,$$

for some constant C_α depending only on α but independent of λ and k.

We define $\eta_k(x) \stackrel{\text{def}}{=} \zeta_k(x)/\sum_k \zeta_k(x)^2$, then $\eta_k(x)$ supports on Ω_k and satisfies

$$\sum_k \zeta_k(x)\eta_k(x) = 1, \qquad \left|D^\alpha \eta_k(x)\right| \leqslant C_\alpha \lambda^{-|\alpha|}. \tag{3.12}$$

Now, we extend functions $\varphi^{(0)}$ and $\varphi^{(1)}$ from K_r to $\mathbb{R}^3_+$ and $\varphi^{(2)}$ from K_r to $\mathbb{R}^3$ with preservation of their classes (for simplicity, we still denote by $\varphi^{(j)}$), and

$$\operatorname{supp}\varphi^{(0)}, \operatorname{supp}\varphi^{(1)} \subset \left\{z \in \mathbb{R}^3_+ \;\middle|\; |z| \leqslant 3r\right\},$$
$$\operatorname{supp}\varphi^{(2)} \subset \left\{z \in \mathbb{R}^3 \;\middle|\; |z| \leqslant 3r\right\},$$

these extended functions have the following properties:

$$\varphi^{(j)}(0) = 0, \qquad \nabla\varphi^{(j)}(0) = 0, \quad j = 0, 1, 2,$$
$$\left\|\varphi^{(j)}\right\|_{W_2^{l+2}(\mathbb{R}^3_+)} \leqslant c\left\|\varphi^{(j)}\right\|_{W_2^{l+3/2}(K_r)} \leqslant C_2(r)M, \quad j = 0, 1,$$
$$\left\|\varphi^{(2)}\right\|_{W_2^{l+2}(\mathbb{R}^3)} \leqslant c\left\|\varphi^{(2)}\right\|_{W_2^{l+3/2}(K_r)} \leqslant C_2(r)M,$$
$$\left|\varphi^{(j)}(z)\right| \leqslant C_1 r \sup_{y' \in K_r}\left|\nabla\varphi^{(j)}\right|, \quad j = 0, 1, 2. \tag{3.13}$$

Furthermore, we get by (3.13) that

$$\sup_{z\in\mathbb{R}^3_+}\left|\nabla\varphi^{(0)}(z)\right|, \sup_{z\in\mathbb{R}^3_+}\left|\nabla\varphi^{(1)}(z)\right|, \sup_{z\in\mathbb{R}^3}\left|\nabla\varphi^{(2)}(z)\right| \leqslant C_3 M r^\beta,$$
$$\sup_{|z|\leqslant\lambda}\left|\nabla\varphi^{(j)}(z)\right| \leqslant C_3 M\lambda^\beta, \quad j = 0, 1, 2, \tag{3.14}$$

where $\beta = \min\{l - 1/2, 1\}$ if $l \neq 3/2$ and $\beta = 1 - \epsilon$ for any $\epsilon \in (0, 1)$ if $l = 3/2$. We shall choose r small enough so that

$$C_3 M r^\beta < 1. \tag{3.15}$$

Here C_1, C_3 are r independent constants, but $C_2(r)$ may depend on r.

Now, we introduce coordinates transformation $z = Z_k(y)$ on $\mathbb{R}^3_+$ so that

$$Z_k^{-1}: \quad y_1 = z_1, \quad y_2 = z_2, \quad y_3 = z_3 + \varphi^{(i)}(z), \quad k = k'', \quad i = 1 \quad \text{or} \quad k = k^{(4)}, \quad i = 0.$$

While on $\mathbb{R}^3$, we introduce coordinates transformation $z = Z_k(y)$ so that

$$Z_k^{-1}: \quad y_1 = z_1, \quad y_2 = z_2, \quad y_3 = z_3 + \varphi^{(2)}(z), \quad k = k'''.$$

Thanks to (3.14) and (3.15), Z_k is invertible, and we denote $J_k(y) \stackrel{\text{def}}{=} (\frac{\partial Z_k(y)}{\partial y})$, then

$$J_{k''}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \varphi_{z_1}^{(1)} & \varphi_{z_2}^{(1)} & 1 + \varphi_{z_3}^{(1)} \end{pmatrix}.$$

Similar expressions for $J_{k'''}^{-1}$ and $J_{k^{(4)}}^{-1}$.

Before proceeding further, we evoke the following results from [15,21]:

Lemma 3.1. *(See [15, Theorem 3.2].) Let $l > 1/2$, $\gamma > 0$, $T > 0$ and $\sigma \geqslant 0$, $\nu > 0$. Let $f \in H_\gamma^{l,l/2}(\mathbb{D}_T)$ ($\mathbb{D}_T \stackrel{\text{def}}{=} \mathbb{R}_+^3 \times (0,T)$), $b_\alpha \in H_\gamma^{l+1/2,l/2+1/4}(\mathbb{R}_T^2)$, $\alpha = 1,2$, $b_3' \in H_\gamma^{l+1/2,1/2,l/2}(\mathbb{R}_T^2)$, $B \in H_\gamma^{l-1/2,l/2-1/4}(\mathbb{R}_T^2)$. Let $g = \nabla \cdot R \in H_\gamma^{l+1,l/2+1/2}(\mathbb{D}_T)$ with $R \in H_\gamma^{0,l/2+1}(\mathbb{D}_T)$. Then the initial–boundary-value problem*

$$\begin{cases} w_t - \nu \nabla^2 w + \nabla p = f, \quad \nabla \cdot w = g \quad \text{in } \mathbb{D}_T, \\ w|_{t=0} = 0 \quad \text{on } \mathbb{R}_+^3, \\ \nu\left(\dfrac{\partial w_3}{\partial z_\alpha} + \dfrac{\partial w_\alpha}{\partial z_3}\right)\Big|_{z_3=0} = b_\alpha \quad \text{on } \mathbb{R}_T^2,\ \alpha = 1,2, \\ -p + 2\nu \dfrac{\partial w_3}{\partial z_3} + \sigma \displaystyle\int_0^t \nabla'^2 w_3\, d\tau \Big|_{z_3=0} = b_3' + \sigma \int_0^t B\, d\tau \quad \text{on } \mathbb{R}_T^2, \end{cases} \tag{3.16}$$

has a unique solution (w,p) with $w \in H_\gamma^{l+2,l/2+1}(\mathbb{D}_T)$, $\nabla p \in H_\gamma^{l,l/2}(\mathbb{D}_T)$ for $\gamma \geqslant \gamma_0 \gg 1$, and there holds

$$\begin{aligned} &\|w\|_{H_\gamma^{l+2,l/2+1}(\mathbb{D}_T)} + \|\nabla p\|_{H_\gamma^{l,l/2}(\mathbb{D}_T)} \\ &\quad \leqslant C\Bigg(\|f\|_{H_\gamma^{l,l/2}(\mathbb{D}_T)} + \|g\|_{H_\gamma^{l+1,0}(\mathbb{D}_T)} + \|R\|_{H_\gamma^{0,l/2+1}(\mathbb{D}_T)} + \big\|b_3'\big\|_{H_\gamma^{l+1/2,1/2,l/2}(\mathbb{R}_T^2)} \\ &\qquad + \sum_{\alpha=1}^2 \|b_\alpha\|_{H_\gamma^{l+1/2,l/2+1/4}(\mathbb{R}_T^2)} + \sigma \|B\|_{H_\gamma^{l-1/2,l/2-1/4}(\mathbb{R}_T^2)}\Bigg), \end{aligned} \tag{3.17}$$

where $\nabla' = {}^t(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2})$.

Lemma 3.2. *(See [21, Theorem 3.5].) Let $l > 1/2$, $\gamma > 0$, $T > 0$ and $\sigma \geqslant 0$, $\nu^{(j)} > 0$, $j = 1,2$. Let $f^{(j)} \in H_\gamma^{l,l/2}(\mathbb{D}_T^{(j)})$ (here $\mathbb{D}_T^{(1)} \stackrel{\text{def}}{=} \mathbb{R}_+^3 \times (0,T)$ and $\mathbb{D}_T^{(2)} \stackrel{\text{def}}{=} \mathbb{R}_-^3 \times (0,T)$), $b_\alpha \in H_\gamma^{l+1/2,l/2+1/4}(\mathbb{R}_T^2)$, $\alpha = 1,2$, $b_3' \in H_\gamma^{l+1/2,1/2,l/2}(\mathbb{R}_T^2)$, $B \in H_\gamma^{l-1/2,l/2-1/4}(\mathbb{R}_T^2)$. Let $g^{(j)} = \nabla \cdot R^{(j)} \in H_\gamma^{l+1,l/2+1/2}(\mathbb{D}_T^{(j)})$ with $R^{(j)} \in H_\gamma^{0,l/2+1}(\mathbb{D}_T^{(j)})$, $\bar\rho^{(1)}, \bar\rho^{(2)}$ be two positive constants. Then the initial–boundary-value problem*

$$\begin{cases} \bar\rho^{(1)} w_t^{(1)} - \nu^{(1)} \nabla^2 w^{(1)} + \nabla p^{(1)} = f^{(1)}, \quad \nabla \cdot w^{(1)} = g^{(1)} \quad \text{in } \mathbb{D}_T^{(1)}, \\ \bar\rho^{(2)} w_t^{(2)} - \nu^{(2)} \nabla^2 w^{(2)} + \nabla p^{(2)} = f^{(2)}, \quad \nabla \cdot w^{(2)} = g^{(2)} \quad \text{in } \mathbb{D}_T^{(2)}, \\ w^{(1)}\big|_{t=0} = 0 \quad \text{on } \mathbb{R}_+^3, \quad w^{(2)}\big|_{t=0} = 0 \quad \text{on } \mathbb{R}_-^3, \end{cases} \tag{3.18}$$

together with the boundary conditions

$$\begin{cases} w^{(1)} - w^{(2)}\big|_{z_3=0} = 0 \quad on\ \mathbb{R}_T^2, \\ \nu^{(2)}\left(\dfrac{\partial w_3^{(2)}}{\partial z_\alpha} + \dfrac{\partial w_\alpha^{(2)}}{\partial z_3}\right) - \nu^{(1)}\left(\dfrac{\partial w_3^{(1)}}{\partial z_\alpha} + \dfrac{\partial w_\alpha^{(1)}}{\partial z_3}\right)\Big|_{z_3=0} = b_\alpha \quad on\ \mathbb{R}_T^2,\ \alpha = 1, 2, \\ p^{(1)} - p^{(2)} + 2\nu^{(2)}\dfrac{\partial w_3^{(2)}}{\partial z_3} - 2\nu^{(1)}\dfrac{\partial w_3^{(1)}}{\partial z_3} \\ \quad - \dfrac{\sigma}{2}\displaystyle\int_0^t \nabla'^2\big(w_3^{(1)} + w_3^{(2)}\big)\,d\tau\Big|_{z_3=0} = b_3' + \dfrac{\sigma}{2}\int_0^t B\,d\tau \quad on\ \mathbb{R}_T^2, \end{cases} \tag{3.19}$$

has a unique solution $(w^{(1)}, p^{(1)}, w^{(2)}, p^{(2)})$ *with* $w^{(j)} \in H_\gamma^{l+2,l/2+1}(\mathbb{D}_T^{(j)})$, $\nabla p^{(j)} \in H_\gamma^{l,l/2}(\mathbb{D}_T^{(j)})$, *and there holds*

$$\begin{aligned} &\sum_{j=1}^{2}\big(\|w^{(j)}\|_{H_\gamma^{l+2,l/2+1}(\mathbb{D}_T^{(j)})} + \|\nabla p^{(j)}\|_{H_\gamma^{l,l/2}(\mathbb{D}_T^{(j)})}\big) \\ &\quad \leqslant C\Bigg[\sum_{j=1}^{2}\big(\|f^{(j)}\|_{H_\gamma^{l,l/2}(\mathbb{D}_T^{(j)})} + \|g^{(j)}\|_{H_\gamma^{l+1,0}(\mathbb{D}_T^{(j)})} + \|R^{(j)}\|_{H_\gamma^{0,l/2+1}(\mathbb{D}_T^{(j)})}\big) \\ &\qquad + \sum_{\alpha=1}^{2}\|b_\alpha\|_{H_\gamma^{l+1/2,l/2+1/4}(\mathbb{R}_T^2)} + \|b_3'\|_{H_\gamma^{l+1/2,1/2,l/2}(\mathbb{R}_T^2)} + \sigma\|B\|_{H_\gamma^{l-1/2,l/2-1/4}(\mathbb{R}_T^2)}\Bigg]. \end{aligned} \tag{3.20}$$

Remark 3.1. In fact, the existence of (3.18)–(3.19) was established under more general situations in [21].

Now we are in a position to prove Theorem 3.3.

Proof of Theorem 3.3. Since the proof of Theorem 3.3 is a little bit long, we will divide it into several steps.

Step 1. We define

$$\mathscr{L}_1 D \stackrel{\text{def}}{=} \sum_{k''}\big(\eta_{k''}u^{(11)(k'')}, \eta_{k''}q^{(11)(k'')}, 0, 0\big) = \big(u^{(11)}, q^{(11)}, 0, 0\big) \quad \text{and}$$

$$\begin{aligned} u^{(11)(k'')}(x,t) &\stackrel{\text{def}}{=} {}^tL_{k''}w^{(k'')}\big(Z_{k''}\circ Y_{k''}(x), t\big), \\ q^{(11)(k'')}(x,t) &\stackrel{\text{def}}{=} s^{(k'')}\big(Z_{k''}\circ Y_{k''}(x), t\big), \end{aligned} \tag{3.21}$$

with $(w^{(k'')}, s^{(k'')})$ satisfying

$$\begin{cases} \rho_0^{(1)}\big(\xi^{(k'')}\big)w_t^{(k'')} - \nu^{(1)}\nabla_{k''}^2 w^{(k'')} + \nabla_{k''}s^{(k'')} = 0, \quad \nabla_{k''}\cdot w^{(k'')} = 0 \quad \text{in } \mathbb{D}_T, \\ w^{(k'')}\big|_{t=0} = 0 \quad \text{on } \mathbb{R}_+^3, \\ \nu^{(1)}\tilde{\Pi}_{k''}\big[\mathrm{D}_{k''}\big(w^{(k'')}\big)e_3\big]\big|_{z_3=0} = 0 \quad \text{on } \mathbb{R}_T^2, \\ -s^{(k'')} + 2\nu^{(1)}e_3\cdot\dfrac{\partial w^{(k'')}}{\partial e_3} - \sigma^{(1)}e_3\cdot\Delta_{k''}^{(1)}\displaystyle\int_0^t w^{(k'')}\,d\tau\Big|_{z_3=0} \\ \quad = \sigma^{(1)}\displaystyle\int_0^t \big(\zeta_{k''}D^{(1)}\big)\big(Y_{k''}^{-1}\circ Z_{k''}^{-1}(z)\big)\,d\tau \quad \text{on } \mathbb{R}_T^2, \end{cases} \tag{3.22}$$

where $\nabla_{k''} = \nabla_z$, $e_3 = (0,0,-1)$, $\mathrm{D}_{k''}(w^{(k'')}) = \nabla_{k''}w^{(k'')} + {}^t\nabla_{k''}w^{(k'')}$, $\tilde{\Pi}_{k''}f(z) = f(z) - (e_3\cdot f(z))e_3$, and $\Delta_{k''}^{(1)} \overset{\text{def}}{=} \nabla_{k''}'^2$ is the Laplace–Beltrami operator on $T_{k''}S_F^{(1)}$ the tangential plane of $S_F^{(1)}$ at $\xi^{(k'')}$.

Let $\Pi_{k''}f \overset{\text{def}}{=} f - (f\cdot n_{k''})n_{k''}$, $n_{k''} \overset{\text{def}}{=} N^{(1)}(\xi^{(k'')})$ and $\bar{\mathrm{D}}_x^{(k'')}(v) \overset{\text{def}}{=} {}^tJ_{k''}^{-1}L_{k''}\nabla_x v + {}^t({}^tJ_{k''}^{-1}L_{k''}\nabla_x v)$. Note that $e_3 = L_{k''}n_{k''}$, thanks to (3.21) and (3.22), we get by using changes of variables from z to x that

$$\begin{cases} \rho_0^{(1)}u_t^{(11)} - \nu^{(1)}\nabla^2 u^{(11)} + \nabla q^{(11)} = \bar{f}^{(1)} \quad \text{in } Q_T^{(1)}, \\ \nabla\cdot u^{(11)} = \bar{g}^{(1)} \quad \text{in } Q_T^{(1)}, \\ \nu^{(1)}\Pi^{(1)}\big[\mathrm{D}\big(u^{(11)}\big)N^{(1)}\big]\big|_{S_{F,T}^{(1)}} = \vec{d}^{(1)}, \\ -q^{(11)} + 2\nu^{(1)}N^{(1)}\cdot\big(\nabla u^{(11)}N^{(1)}\big) - \sigma^{(1)}N^{(1)}\cdot\Delta^{(1)}\displaystyle\int_0^t u^{(11)}\,d\tau\Big|_{S_{F,T}^{(1)}} \\ \quad = d^{(1)} + \sigma^{(1)}\displaystyle\int_0^t \big(D^{(1)} + H^{(1)}\big)\,d\tau, \end{cases} \tag{3.23}$$

where

$$\begin{aligned} \bar{f}^{(1)} = {} & -\nu^{(1)}\sum_{k''}\big[\nabla^2\big(\eta_{k''}u^{(11)(k'')}\big) - \eta_{k''}\big({}^tJ_{k''}^{-1}L_{k''}\nabla\big)^2 u^{(11)(k'')}\big] \\ & + \sum_{k''}\big[\nabla\big(\eta_{k''}q^{(11)(k'')}\big) - \eta_{k''}{}^tL_{k''}{}^tJ_{k''}^{-1}L_{k''}\nabla q^{(11)(k'')}\big] \\ & + \sum_{k''}\big[\eta_{k''}\big(\rho_0^{(1)}(x) - \rho_0^{(1)}\big(\xi^{(k'')}\big)\big)u_t^{(11)(k'')}\big], \end{aligned}$$

and

$$\begin{aligned} \bar{g}^{(1)} &= \sum_{k''}\big[\nabla\eta_{k''}\cdot u^{(11)(k'')} + \eta_{k''}\nabla\cdot u^{(11)(k'')}\big] \overset{\text{def}}{=} \nabla\cdot\bar{R}^{(1)}, \\ \vec{d}^{(1)} &= \nu^{(1)}\sum_{k''}\Pi^{(1)}\big[\mathrm{D}\big(\eta_{k''}u^{(11)(k'')}\big)N^{(1)} - \eta_{k''}\Pi_{k''}\big[{}^tL_{k''}\bar{\mathrm{D}}_x^{(k'')}\big(L_{k''}u^{(11)(k'')}\big)L_{k''}n_{k''}\big]\big], \\ d^{(1)} &= 2\nu^{(1)}\sum_{k''}\big[N^{(1)}\cdot\big(\nabla\big(\eta_{k''}u^{(11)(k'')}\big)N^{(1)}\big) - \eta_{k''}n_{k''}\cdot\big({}^tL_{k''}{}^tJ_{k''}^{-1}L_{k''}\nabla u^{(11)(k'')}n_{k''}\big)\big], \end{aligned}$$

$$H^{(1)} = -\sum_{k''} \bigl[N^{(1)} \cdot \Delta^{(1)}\bigl(\eta_{k''} u^{(11)(k'')}\bigr) - \eta_{k''} n_{k''} \cdot \Delta^{(1)}_{k''} u^{(11)(k'')}\bigr].$$

Thanks to Lemma 3.1, (3.22) has a unique solution $(w^{(k'')}, s^{(k'')})$ such that $w^{(k'')} \in H_\gamma^{l+2,l/2+1}(\mathbb{D}_T)$, $\nabla s^{(k'')} \in H_\gamma^{l,l/2}(\mathbb{D}_T)$ and

$$\bigl\| w^{(k'')}\bigr\|_{H_\gamma^{l+2,l/2+1}(\mathbb{D}_T)} + \bigl\|\nabla s^{(k'')}\bigr\|_{H_\gamma^{l,l/2}(\mathbb{D}_T)} \leqslant C\sigma^{(1)} \bigl\| \bigl(\zeta_{k''} D^{(1)}\bigr)\bigl(Y_{k''}^{-1} \circ Z_{k''}^{-1}(z)\bigr)\bigr\|_{H_\gamma^{l-1/2,l/2-1/4}(\mathbb{R}^2_T)},$$

which ensures that

$$\bigl\|u^{(11)(k'')}\bigr\|_{H_\gamma^{l+2,l/2+1}(Q_T^{(k'')})} + \bigl\|\nabla q^{(11)(k'')}\bigr\|_{H_\gamma^{l,l/2}(Q_T^{(k'')})} \leqslant C\sigma^{(1)} \bigl\|\zeta_{k''} D^{(1)}\bigr\|_{H_\gamma^{l-1/2,l/2-1/4}(Q_T^{(1)})}, \tag{3.24}$$

where $Q_T^{(k'')} \stackrel{\text{def}}{=} \Omega_{k''} \times (0,T)$.

Now let us turn to the estimate of $\bar f^{(1)}$, $\bar g^{(1)}$, $\vec d^{(1)}$, $d^{(1)}$, and $H^{(1)}$. First of all, we get by (3.13)–(3.15) that

$$\begin{aligned}
&\sup_{\tilde\Omega_{k''}} \bigl|N^{(1)}(z) - n_{k''}\bigr| \leqslant C\lambda^\beta, \qquad \bigl\|N^{(1)}(z) - n_{k''}\bigr\|_{W_2^{l+1}(\tilde\Omega_{k''})} \leqslant C,\\
&\sup_{\tilde\Omega_{k''}} |I - J_{k''}|, \sup_{\tilde\Omega_{k''}} \bigl|I - J_{k''}^{-1}\bigr| \leqslant C\lambda^\beta, \qquad \|J_{k''}\|_{W_2^{l+1}(\tilde\Omega_{k''})}, \bigl\|J_{k''}^{-1}\bigr\|_{W_2^{l+1}(\tilde\Omega_{k''})} \leqslant C,
\end{aligned} \tag{3.25}$$

where $\tilde\Omega_{k''} \stackrel{\text{def}}{=} \{z \in \mathbb{R}^3_+ \mid Y_{k''}^{-1} \circ Z_{k''}^{-1}(z) \in \Omega_{k''}\}$. Moreover,

$$\sum_{k''} \|\nabla J_{k''}\|^2_{W_2^l(\tilde\Omega_{k''})}, \sum_{k''} \bigl\|\nabla J_{k''}^{-1}\bigr\|^2_{W_2^l(\tilde\Omega_{k''})} \leqslant C, \tag{3.26}$$

since $\varphi^{(1)}$ can be represented in terms of $F_0^{(1)}$ in the neighborhood of $\xi^{(k'')} \in S_F^{(1)}$.

Step 1.1. The estimate of $\bar f^{(1)}$.

From (3.25) and Lemma 2.1, we deduce that

$$\begin{aligned}
&\bigl\|\nabla\bigl(\eta_{k''} q^{(11)(k'')}\bigr) - \eta_{k''} {}^tL_{k''} {}^tJ_{k''}^{-1} L_{k''} \nabla q^{(11)(k'')}\bigr\|_{H_\gamma^{l,l/2}(Q_T^{(1)})}\\
&\quad = \bigl\|\nabla\eta_{k''} q^{(11)(k'')} + \eta_{k''}\bigl(I - {}^tL_{k''} {}^tJ_{k''}^{-1} L_{k''}\bigr)\nabla q^{(11)(k'')}\bigr\|_{H_\gamma^{l,l/2}(Q_T^{(1)})}\\
&\quad \leqslant C(\lambda)\bigl\|q^{(11)(k'')}\bigr\|_{H_\gamma^{l,l/2}(Q_T^{(k'')})} + \bigl\|\nabla q^{(11)(k'')}\bigr\|_{H_\gamma^{l,l/2}(Q_T^{(k'')})}\bigl[C \sup\bigl|\eta_{k''}\bigl(I - {}^tL_{k''} {}^tJ_{k''}^{-1} L_{k''}\bigr)\bigr|\\
&\qquad + \bigl(\epsilon + C(\epsilon)\gamma^{-l/2}\bigr)\bigl\|\eta_{k''}\bigl(I - {}^tL_{k''} {}^tJ_{k''}^{-1} L_{k''}\bigr)\bigr\|_{W_2^{l+1}(\Omega^{(1)})}\bigr]\\
&\quad \leqslant \bigl(C\lambda^\beta + C(\lambda)\gamma^{-l/2}\bigr)\bigl\|\nabla q^{(11)(k'')}\bigr\|_{H_\gamma^{l,l/2}(Q_T^{(k'')})} + C(\lambda)\bigl\|q^{(11)(k'')}\bigr\|_{H_\gamma^{l,l/2}(Q_T^{(k'')})},
\end{aligned}$$

by taking ϵ small enough depending only on λ in the last inequality. Similarly, we have

$$\begin{aligned}
&\bigl\|\eta_{k''}\bigl(\rho_0^{(1)}(x) - \rho_0^{(1)}\bigl(\xi^{(k'')}\bigr)\bigr)u_t^{(11)(k'')}\bigr\|_{H_\gamma^{l,l/2}(Q_T^{(1)})}\\
&\quad \leqslant \bigl(C\lambda^\beta + C(\lambda)\gamma^{-l/2}\bigr)\bigl\|u^{(11)(k'')}\bigr\|_{H_\gamma^{l+2,l/2+1}(Q_T^{(k'')})},
\end{aligned}$$

where we used (2.1) and the fact that

$$\big|\rho_0^{(1)}(x)-\rho_0^{(1)}\big(\xi^{(k'')}\big)\big| \leqslant \lambda^\beta \sup_{\Omega^{(1)}} \frac{|\rho_0^{(1)}(x)-\rho_0^{(1)}(\xi^{(k'')})|}{|x-\xi^{(k'')}|^\beta} \leqslant C\lambda^\beta\big(1+\big\|\nabla\rho_0^{(1)}\big\|_{W_2^l(\Omega^{(1)})}\big) \leqslant C\lambda^\beta.$$

A simple calculation gives

$$\begin{aligned}\nabla^2\big(\eta_{k''}u^{(11)(k'')}\big)-\eta_{k''}\big({}^tJ_{k''}^{-1}L_{k''}\nabla\big)^2u^{(11)(k'')} &= \nabla^2\eta_{k''}u^{(11)(k'')}+2(\nabla\eta_{k''}\cdot\nabla)u^{(11)(k'')}\\ &\quad+\eta_{k''}\big(I-{}^tL_{k''}{}^tJ_{k''}^{-1}L_{k''}\big)\nabla\cdot\nabla u^{(11)(k'')}\\ &\quad+\eta_{k''}{}^tL_{k''}{}^tJ_{k''}^{-1}L_{k''}\nabla\cdot\big(\big(I-{}^tL_{k''}{}^tJ_{k''}^{-1}L_{k''}\big)\nabla u^{(11)(k'')}\big).\end{aligned}$$

We get by Lemma 2.1 and (2.2) that

$$\begin{aligned}\big\|\nabla^2\eta_{k''}u^{(11)(k'')}\big\|_{H_\gamma^{l,l/2}(Q_T^{(k'')})} &\leqslant C(\lambda)\big\|u^{(11)(k'')}\big\|_{H_\gamma^{l,l/2}(Q_T^{(k'')})}\\ &\leqslant C(\lambda)\big(\epsilon+C\gamma^{-l/2-1}\epsilon^{-\frac{l}{2}}\big)\big\|u^{(11)(k'')}\big\|_{H_\gamma^{l+2,l/2+1}(Q_T^{(k'')})}\\ &\leqslant \big(C\lambda^\beta+C(\lambda)\gamma^{-l/2-1}\big)\big\|u^{(11)(k'')}\big\|_{H_\gamma^{l+2,l/2+1}(Q_T^{(k'')})}.\end{aligned}$$

Similar calculation applied to the other terms, we finally obtain

$$\begin{aligned}&\big\|\nabla^2\big(\eta_{k''}u^{(11)(k'')}\big)-\eta_{k''}\big({}^tJ_{k''}^{-1}L_{k''}\nabla\big)^2u^{(11)(k'')}\big\|_{H_\gamma^{l,l/2}(Q_T^{(1)})}\\ &\quad\leqslant \big(C\lambda^\beta+C(\lambda)\gamma^{-l/2}\big)\big\|u^{(11)(k'')}\big\|_{H_\gamma^{l+2,l/2+1}(Q_T^{(k'')})}.\end{aligned}$$

Therefore, thanks to (3.24), we obtain

$$\begin{aligned}\big\|\bar f^{(1)}\big\|^2_{H_\gamma^{l,l/2}(Q_T^{(1)})} &\leqslant \big(C\lambda^{2\beta}+C(\lambda)\gamma^{-l}\big)\sigma^{(1)2}\sum_{k''}\big\|\zeta_{k''}D^{(1)}\big\|^2_{H_\gamma^{l-1/2,l/2-1/4}(S_{F,T}^{(1)})}\\ &\quad+C(\lambda)\sum_{k''}\big\|q^{(11)(k'')}\big\|^2_{H_\gamma^{l,l/2}(Q_T^{(k'')})}.\end{aligned}\tag{3.27}$$

This reduces the estimate of $\|\bar f^{(1)}\|_{H_\gamma^{l,l/2}(Q_T^{(1)})}$ to that of $\|q^{(11)(k'')}\|_{H_\gamma^{l,l/2}(Q_T^{(k'')})}$ or $\|s^{(k'')}\|_{H_\gamma^{l,l/2}(\mathbb{D}_T)}$. Thanks to its definition, we have

$$\begin{aligned}\big\|s^{(k'')}\big\|^2_{H_\gamma^{l,0}(\mathbb{D}_T)} &\leqslant \int_0^T e^{-2\gamma t}\big[\epsilon\big\|\nabla s^{(k'')}\big\|^2_{\dot W_2^l(\mathbb{R}^3_+)}+C(\epsilon)\big\|s^{(k'')}\big\|^2_{L^2(\mathbb{R}^3_+)}\big]\,dt\\ &\leqslant \epsilon\big\|\nabla s^{(k'')}\big\|^2_{H_\gamma^{l,0}(\mathbb{D}_T)}+C(\epsilon)\gamma^{-l}\big\|s^{(k'')}\big\|^2_{H_\gamma^{0,l/2}(\mathbb{D}_T)}.\end{aligned}$$

So, it suffices to estimate $\|s^{(k'')}\|_{H_\gamma^{0,l/2}(\mathbb{D}_T)}$. We get by taking divergence to the first equation of (3.22) that (see also Section 5 of [15])

$$\begin{cases} \nabla_{k''}^2 s^{(k'')} = 0 \quad \text{in } \mathbb{R}_+^3, \\ s^{(k'')}\big|_{z_3=0} = 2\nu^{(1)} \dfrac{\partial w_3^{(k'')}}{\partial z_3} + \sigma^{(1)} \displaystyle\int_0^t \left[\nabla_{k''}'^2 w_3^{(k'')} - \left(\zeta_{k''} D^{(1)}\right)\left(Y_{k''}^{-1} \circ Z_{k''}^{-1}(z)\right)\right] d\tau \bigg|_{z_3=0} \\ \qquad \overset{\text{def}}{=} \psi_{k''} + \sigma^{(1)} \displaystyle\int_0^t \phi_{k''}\, d\tau \quad \text{on } \mathbb{R}^2. \end{cases}$$

Then we first extend $\psi_{k''}$, $\phi_{k''}$ from $(0,T)$ to $(0,\infty)$ with preservation of the classes (see Section 2 of [15]) and then apply Laplace transformation

$$\tilde{f}(z,\zeta) = \int_0^\infty e^{-\zeta t} f(z,t)\, dt$$

to the above system to obtain

$$\begin{cases} \nabla_{k''}^2 \tilde{s}^{(k'')} = 0 \quad \text{in } \mathbb{R}_+^3, \\ \tilde{s}^{(k'')}\big|_{z_3=0} = \tilde{\psi}_{k''} + \dfrac{\sigma^{(1)}}{\zeta} \tilde{\phi}_{k''} \quad \text{on } \mathbb{R}^2, \end{cases}$$

which gives

$$\tilde{s}^{(k'')}(z,\zeta) = \frac{z_3}{2\pi} \int_{\mathbb{R}^2} \frac{\tilde{\psi}_{k''}(y',\zeta) + \frac{\sigma^{(1)}}{\zeta} \tilde{\phi}_{k''}(y',\zeta)}{|(z',z_3) - (y',0)|^3}\, dy' \quad \text{for } z \in \mathbb{R}_+^3.$$

Let $K_{m\lambda}^{(k'')} \overset{\text{def}}{=} \{z \in \mathbb{R}_+^3 \mid |z| \leqslant m\lambda\}$ related to $\xi^{(k'')}$, from the Young's inequality and an equivalent definition of the norms (see [15, Lemma 2.1]), we infer that

$$\begin{aligned} &\|s_{k''}\|^2_{H_\gamma^{0,l/2+1/4}(K_{m\lambda}^{(k'')} \times (0,T))} \\ &\quad \leqslant C\left\{\left(\lambda + \gamma^{-1}\right)\|w^{(k'')}\|^2_{H_\gamma^{l+2,l/2+1}(\mathbb{D}_T)} + {\sigma^{(1)}}^2 \gamma^{-1} \left\|\left(\zeta_{k''} D^{(1)}\right)\left(Y_{k''}^{-1} \circ Z_{k''}^{-1}(z)\right)\right\|^2_{H_\gamma^{0,l/2-1/4}(\mathbb{R}_T^2)}\right\}, \end{aligned}$$

which ensures

$$\|q^{(11)(k'')}\|_{H_\gamma^{l,l/2}(Q_T^{(k'')})} \leqslant \mu_1(\lambda,\gamma)\sigma^{(1)} \|\zeta_{k''} D^{(1)}\|_{H_\gamma^{l-1/2,l/2-1/4}(S_{F,T}^{(1)})},$$

where $\mu_1(\lambda,\gamma) \to 0$ as $\lambda \to 0$ and then $\gamma \to +\infty$. In what follows, we shall denote by $\mu_i(\lambda,\gamma)$ the constants of this type. Then thanks to (3.27), we arrive at

$$\begin{aligned} \|\bar{f}^{(1)}\|_{H_\gamma^{l,l/2}(Q_T^{(1)})} &\leqslant \mu_2(\lambda,\gamma)\sigma^{(1)} \left(\sum_{k''} \|\zeta_{k''} D^{(1)}\|^2_{H_\gamma^{l-1/2,l/2-1/4}(S_{F,T}^{(1)})}\right)^{\frac{1}{2}} \\ &\leqslant \mu_3(\lambda,\gamma)\sigma^{(1)} \|D^{(1)}\|_{H_\gamma^{l-1/2,l/2-1/4}(S_{F,T}^{(1)})}. \end{aligned} \tag{3.28}$$

Step 1.2. The estimates of $\bar{g}^{(1)}$ and $\bar{R}^{(1)}$.

We reformulate $\bar{g}^{(1)}$ as

$$\bar{g}^{(1)} = \sum_{k''} \nabla \cdot \left[\eta_{k''}\left(I - {}^t L_{k''} J_{k''}^{-1} L_{k''}\right) u^{(11)(k'')}\right] + \sum_{k''} \left(\nabla \cdot \left(\eta_{k''} {}^t L_{k''} J_{k''}^{-1} L_{k''}\right)\right) \cdot u^{(11)(k'')}$$
$$\stackrel{\text{def}}{=} \nabla \cdot \bar{R}^{(1)},$$

with

$$\begin{aligned}\bar{R}^{(1)} &= \sum_{k''} \eta_{k''}\left(I - {}^t L_{k''} J_{k''}^{-1} L_{k''}\right) u^{(11)(k'')} \\ &\quad - \sum_{k''} \frac{1}{4\pi} \nabla \int_{\Omega^{(1)}} \frac{1}{|x-y|} \sum_{k''} \left(\nabla \eta_{k''} \cdot \left({}^t L_{k''} J_{k''}^{-1} L_{k''}\right)\right) \cdot u^{(11)(k'')}\, dy \\ &\quad - \sum_{k''} \frac{1}{4\pi} \nabla \int_{\Omega^{(1)}} \frac{1}{|x-y|} \sum_{k''} \eta_{k''} \left(L_{k''} \nabla \cdot \left(J_{k''}^{-1} L_{k''}\right)\right) \cdot u^{(11)(k'')}\, dy \\ &\stackrel{\text{def}}{=} \bar{R_1}^{(1)} + \bar{R_2}^{(1)} + \bar{R_3}^{(1)}.\end{aligned}$$

We first get by a similar calculation as that in Step 1.1 that

$$\left\|\bar{g}^{(1)}\right\|_{H_\gamma^{l+1,l/2+1/2}(Q_T^{(1)})} \leqslant \mu_4(\lambda,\gamma)\sigma^{(1)} \left\|D^{(1)}\right\|_{H_\gamma^{l-1/2,l/2-1/4}(S_{F,T}^{(1)})}.$$

Then it remains to handle $\bar{R}^{(1)}$. Firstly thanks to (3.25), we have

$$\left\|\bar{R_1}^{(1)}\right\|^2_{H_\gamma^{0,l/2+1}(Q_T^{(1)})} \leqslant C\lambda^{2\beta} \sum_{k''} \left\|u^{(11)(k'')}\right\|^2_{H_\gamma^{0,l/2+1}(Q_T^{(k'')})}. \tag{3.29}$$

While we get by using potential theory [18] that

$$\begin{aligned}\left\|\bar{R_3}^{(1)}\right\|_{L^2(\Omega^{(1)})} &\leqslant C \sum_{k''} \left\|\eta_{k''}\left(L_{k''} \nabla \cdot \left(J_{k''}^{-1} L_{k''}\right)\right) \cdot u^{(11)(k'')}\right\|_{L^{6/5}(\Omega^{(1)})} \\ &\leqslant C \sum_{k''} \lambda^{3(1/3-1/p)} \left\|\nabla J_{k''}^{-1}\right\|_{L^p(\Omega_{k''})} \left\|u^{(11)(k'')}\right\|_{L^2(\Omega_{k''})} \\ &\leqslant C \sum_{k''} \lambda^{\beta} \left\|\nabla J_{k''}^{-1}\right\|_{W_2^l(\Omega_{k''})} \left\|u^{(11)(k'')}\right\|_{L^2(\Omega_{k''})} \\ &\leqslant C\lambda^{\beta} \left(\sum_{k''} \left\|u^{(11)(k'')}\right\|^2_{L^2(\Omega_{k''})}\right)^{\frac{1}{2}},\end{aligned}$$

where we used Cauchy–Schwartz inequality and (3.26) in the last step, $p > 3$ and $1 - \frac{3}{p} = \min(l - \frac{1}{2}, 1)$. Consequently, we obtain

$$\left\|\bar{R_3}^{(1)}\right\|^2_{H_\gamma^{0,l/2+1}(Q_T^{(1)})} \leqslant C\lambda^{2\beta} \sum_{k''} \left\|u^{(11)(k'')}\right\|^2_{H_\gamma^{0,l/2+1}(Q_T^{(k'')})}. \tag{3.30}$$

To handle $\bar{R}_2^{(1)}$, we first calculate

$$\bar{R_2}_t^{(1)} = -\frac{1}{4\pi}\nabla \int\limits_{\Omega^{(1)}} \frac{1}{|x-y|}\sum_{k''}\big(\nabla\eta_{k''}\cdot\big({}^tL_{k''}J_{k''}^{-1}L_{k''}\big)\big)\cdot u_t^{(11)(k'')}\,dy.$$

We denote $\vec{\phi}_{k''} \overset{\text{def}}{=} \nabla\eta_{k''}\cdot({}^tL_{k''}J_{k''}^{-1}L_{k''}) \in W_2^{l+1}(\Omega_{k''})$ and $\nabla_1 \overset{\text{def}}{=} {}^tJ_{k''}^{-1}L_{k''}\nabla$. Then we get from the equation of $u_t^{(11)(k'')}$ that

$$\begin{aligned}
\vec{\phi}_{k''}\cdot u_t^{(11)(k'')} &= \nabla\cdot\left[\frac{\nu^{(1)}}{\rho_0^{(1)}(\xi^{(k'')})}{}^tL_{k''}J_{k''}^{-1}\nabla_1 u^{(11)(k'')}\vec{\phi}_{k''} - \frac{q^{(11)(k'')}}{\rho_0^{(1)}(\xi^{(k'')})}{}^tL_{k''}J_{k''}^{-1}L_{k''}\vec{\phi}_{k''}\right] \\
&\quad - \frac{\nu^{(1)}}{\rho_0^{(1)}(\xi^{(k'')})}\nabla_1\vec{\phi}_{k''}:\nabla_1 u^{(11)(k'')} - \frac{\nu^{(1)}}{\rho_0^{(1)}(\xi^{(k'')})}\big(\nabla\cdot\big({}^tL_{k''}J_{k''}^{-1}\big)\big)\cdot\big(\nabla_1 u^{(11)(k'')}\vec{\phi}_{k''}\big) \\
&\quad + \frac{q^{(11)(k'')}}{\rho_0^{(1)}(\xi^{(k'')})}\big(\nabla\cdot\big({}^tL_{k''}J_{k''}^{-1}L_{k''}\big)\big)\cdot\vec{\phi}_{k''} + \frac{q^{(11)(k'')}}{\rho_0^{(1)}(\xi^{(k'')})}{}^tL_{k''}{}^tJ_{k''}^{-1}L_{k''}\nabla\cdot\vec{\phi}_{k''},
\end{aligned}$$

from which, we get by using integration by parts and the potential theory [18] that

$$\begin{aligned}
\big\|\bar{R_2}_t^{(1)}\big\|^2_{H_\gamma^{0,l/2}(Q_T^{(1)})} &\leqslant C(\lambda)\sum_{k''}\big(\big\|\nabla u^{(11)(k'')}\big\|^2_{H_\gamma^{0,l/2}(Q_T^{(k'')})} + \big\|q^{(11)(k'')}\big\|^2_{H_\gamma^{0,l/2}(Q_T^{(k'')})}\big) \\
&\quad + C(\lambda)\sum_{k''}\big(\big\|\nabla u^{(11)(k'')}\big\|^2_{H_\gamma^{0,l/2}(S_{F,T}^{(k'')})} + \big\|q^{(11)(k'')}\big\|^2_{H_\gamma^{0,l/2}(S_{F,T}^{(k'')})}\big). \qquad (3.31)
\end{aligned}$$

On the other hand, since

$$\int\limits_0^T e^{-2\gamma t}\big\|\bar{R_2}^{(1)}(\cdot,t)\big\|^2_{L^2(\Omega^{(1)})}\,dt \leqslant \gamma^{-2}\int\limits_0^T e^{-2\gamma t}\big\|\bar{R_{2t}}^{(1)}(\cdot,t)\big\|^2_{L^2(\Omega^{(1)})}\,dt,$$

we thus have

$$\big\|\bar{R_2}^{(1)}\big\|^2_{H_\gamma^{0,l/2+1}(Q_T^{(1)})} \leqslant C\big\|\bar{R_2}_t^{(1)}\big\|^2_{H_\gamma^{0,l/2}(Q_T^{(1)})},$$

which together with (3.24) and (3.29)–(3.31), we get by (2.1) and (2.2) that

$$\big\|\bar{R}^{(1)}\big\|_{H_\gamma^{0,l/2+1}(Q_T^{(1)})} \leqslant \mu_5(\lambda,\gamma)\sigma^{(1)}\big\|D^{(1)}\big\|_{H_\gamma^{l-1/2,l/2-1/4}(S_{F,T}^{(1)})}. \qquad (3.32)$$

Step 1.3. The estimates of $\vec{d}^{(1)}$, $d^{(1)}$ and $H^{(1)}$.

We firstly get by a simple calculation that

$$\begin{aligned}
\vec{d}^{(1)} &= \nu^{(1)}\sum_{k''}\left[\Pi^{(1)}\big(u^{(11)(k'')}\big)\frac{\partial\eta_{k''}}{\partial N^{(1)}} + \big(u^{(11)(k'')}\cdot N^{(1)}\big)\Pi^{(1)}(\nabla\eta_{k''})\right] \\
&\quad + \nu^{(1)}\sum_{k''}\eta_{k''}\Pi^{(1)}\big[\big(I-{}^tL_{k''}{}^tJ_{k''}^{-1}L_{k''}\big)\nabla u^{(11)(k'')} + {}^t\big(\big(I-{}^tL_{k''}{}^tJ_{k''}^{-1}L_{k''}\big)\nabla u^{(11)(k'')}\big)\big]
\end{aligned}$$

$$
\begin{aligned}
&+\nu^{(1)}\sum_{k''}\eta_{k''}\Pi^{(1)}\big[{}^tL_{k''}\bar{\mathrm{D}}_x^{(k'')}\big(L_{k''}u^{(11)(k'')}\big)L_{k''}\big(N^{(1)}-n_{k''}\big)\big]\\
&+\nu^{(1)}\sum_{k''}\eta_{k''}\big({}^tL_{k''}\bar{\mathrm{D}}_x^{(k'')}\big(L_{k''}u^{(11)(k'')}\big)L_{k''}n_{k''}\cdot n_{k''}\big)\Pi^{(1)}\big(n_{k''}-N^{(1)}\big),
\end{aligned}
$$

$$
\begin{aligned}
d^{(1)}={}&2\nu^{(1)}\sum_{k''}\big(u^{(11)(k'')}\cdot N^{(1)}\big)\frac{\partial\eta_{k''}}{\partial N^{(1)}}+2\nu^{(1)}\sum_{k''}\eta_{k''}\big(N^{(1)}-n_{k''}\big)\cdot\nabla u^{(11)(k'')}N^{(1)}\\
&+2\nu^{(1)}\sum_{k''}\eta_{k''}n_{k''}\cdot\big[\big(I-{}^tL_{k''}{}^tJ_{k''}^{-1}L_{k''}\big)\nabla u^{(11)(k'')}N^{(1)}\big]\\
&+2\nu^{(1)}\sum_{k''}\eta_{k''}n_{k''}\cdot\big[\big({}^tL_{k''}{}^tJ_{k''}^{-1}L_{k''}\nabla u^{(11)(k'')}\big)\big(N^{(1)}-n_{k''}\big)\big],
\end{aligned}
$$

$$
\begin{aligned}
H^{(1)}={}&-\sum_{k''}\big[N^{(1)}\cdot\big(\Delta^{(1)}\big(\eta_{k''}u^{(11)(k'')}\big)-\eta_{k''}\Delta^{(1)}u^{(11)(k'')}\big)\\
&+\eta_{k''}\big(N^{(1)}-n_{k''}\big)\cdot\Delta^{(1)}u^{(11)(k'')}+\eta_{k''}n_{k''}\cdot\big(\Delta^{(1)}-\Delta_{k''}^{(1)}\big)u^{(11)(k'')}\big].
\end{aligned}
$$

On the other hand, we deduce from (1.3) that

$$
\Delta^{(1)}\big(\eta_{k''}u^{(11)(k'')}\big)-\eta_{k''}\Delta^{(1)}u^{(11)(k'')}=h_\alpha\frac{\partial}{\partial y_\alpha}u^{(11)(k'')}\big(Y_{k''}^{-1}(y)\big)+du^{(11)(k'')}\big(Y_{k''}^{-1}(y)\big),
$$

$$
\Delta^{(1)}-\Delta_{k''}^{(1)}=C_{\alpha\beta}\frac{\partial^2}{\partial y_\alpha\partial y_\beta}+D_\delta\frac{\partial}{\partial y_\delta},
$$

where $h_\alpha, C_{\alpha\beta}\in W_2^{l+1}(K_{m\lambda}^{(k'')})$, $D_\delta, d\in W_2^{l}(K_{m\lambda}^{(k'')})$ and $\sup_{K_{m\lambda}^{(k'')}}|C_{\alpha\beta}|\leqslant C\lambda^\beta$.

Therefore, with the help of Lemma 2.1, and (2.1)–(2.4), (3.25), we can get the desired estimates for $\vec{d}^{(1)}$, $d^{(1)}$ and $H^{(1)}$. Combining Step 1.1 to Step 1.3, we arrive at

$$
\begin{aligned}
&\big\|\bar f^{(1)}\big\|_{H_\gamma^{l,l/2}(Q_T^{(1)})}+\big\|\bar g^{(1)}\big\|_{H_\gamma^{l+1,l/2+1/2}(Q_T^{(1)})}+\big\|\bar R^{(1)}\big\|_{H_\gamma^{0,l/2+1}(Q_T^{(1)})}\\
&\quad+\big\|\big(\vec{d}^{(1)},d^{(1)}\big)\big\|_{H_\gamma^{l+1/2,l/2+1/4}(S_{F,T}^{(1)})}+\big\|H^{(1)}\big\|_{H_\gamma^{l-1/2,l/2-1/4}(S_{F,T}^{(1)})}\\
&\leqslant\mu_6(\lambda,\gamma)\sigma^{(1)}\big\|D^{(1)}\big\|_{H_\gamma^{l-1/2,l/2-1/4}(S_{F,T}^{(1)})}.
\end{aligned}
\tag{3.33}
$$

Step 2. We define

$$
\begin{aligned}
\mathscr{L}_2D&\overset{\mathrm{def}}{=}\sum_{k'''}\big(\eta_{k'''}u^{(12)(k''')},\eta_{k'''}q^{(12)(k''')},\eta_{k'''}u^{(22)(k''')},\eta_{k'''}q^{(22)(k''')}\big)\\
&=\big(u^{(12)},q^{(12)},u^{(22)},q^{(22)}\big),
\end{aligned}
$$

with

$$
\begin{aligned}
u^{(j2)(k''')}(x,t)&\overset{\mathrm{def}}{=}{}^tL_{k'''}w^{(j)(k''')}\big(Z_{k'''}\circ Y_{k'''}(x),t\big),\\
q^{(j2)(k''')}(x,t)&\overset{\mathrm{def}}{=}s^{(j)(k''')}\big(Z_{k'''}\circ Y_{k'''}(x),t\big),\quad j=1,2,
\end{aligned}
\tag{3.34}
$$

where $(w^{(1)(k''')}, s^{(1)(k''')}, w^{(2)(k''')}, s^{(2)(k''')})$ satisfies

$$\begin{cases} \rho_0^{(j)}\big(\xi^{(k''')}\big) w_t^{(j)(k''')} - \nu^{(j)} \nabla_{k'''}^2 w^{(j)(k''')} + \nabla_{k'''} s^{(j)(k''')} = 0 \quad \text{in } \mathbb{D}_T^{(j)}, \\ \nabla_{k'''} \cdot w^{(j)(k''')} = 0 \quad \text{in } \mathbb{D}_T^{(j)},\ j = 1, 2, \\ w^{(1)(k''')}\big|_{t=0} = 0 \quad \text{on } \mathbb{R}_+^3, \quad w^{(2)(k''')}\big|_{t=0} = 0 \quad \text{on } \mathbb{R}_-^3, \end{cases} \tag{3.35}$$

together with the boundary conditions

$$\begin{cases} w^{(1)(k''')} - w^{(2)(k''')}\big|_{z_3=0} = 0 \quad \text{on } \mathbb{R}_T^2, \\ \tilde{\Pi}_{k'''}\big[\nu^{(2)} \mathrm{D}_{k'''}\big(w^{(2)(k''')}\big) e_3 - \nu^{(1)} \mathrm{D}_{k'''}\big(w^{(1)(k''')}\big) e_3\big]\big|_{z_3=0} = 0 \quad \text{on } \mathbb{R}_T^2, \\ s^{(1)(k''')} - s^{(2)(k''')} + 2\nu^{(2)} \dfrac{\partial w_3^{(2)(k''')}}{\partial z_3} - 2\nu^{(1)} \dfrac{\partial w_3^{(1)(k''')}}{\partial z_3} \\ \qquad - \dfrac{\sigma^{(2)}}{2} e_3 \cdot \displaystyle\int_0^t \Delta_{k'''}^{(2)}\big(w^{(1)(k''')} + w^{(2)(k''')}\big)\, d\tau \Big|_{z_3=0} \\ \quad = \dfrac{\sigma^{(2)}}{2} \displaystyle\int_0^t \big(\zeta_{k'''} D^{(2)}\big)\big(Y_{k'''}^{-1} \circ Z_{k'''}^{-1}(z)\big)\, d\tau \quad \text{on } \mathbb{R}_T^2, \end{cases} \tag{3.36}$$

where $\nabla_{k'''} = \nabla_z$, $e_3 = (0,0,1)$, $\mathrm{D}_{k'''}(w) = \nabla_{k'''} w + {}^t\nabla_{k'''} w$, $\tilde{\Pi}_{k'''} f(z) = f(z) - (e_3 \cdot f(z)) e_3$ and $\Delta_{k'''}^{(2)} = \nabla_{k'''}'^2$ is the Laplace–Beltrami operator on $T_{k'''} S_F^{(2)}$ the tangential plane of $S_F^{(2)}$ at $\xi^{(k''')}$, while $\nabla_{k'''}' = {}^t(\partial_{z_1}, \partial_{z_2})$.

Thanks to (3.35)–(3.36), we find that $(u^{(12)}, q^{(12)}, u^{(22)}, q^{(22)})$ satisfies

$$\begin{cases} \rho_0^{(j)} u_t^{(j2)} - \nu^{(j)} \nabla^2 u^{(j2)} + \nabla q^{(j2)} = f^{(j)}, \quad \nabla \cdot u^{(j2)} = g^{(j)} \quad \text{in } Q_T^{(j)}, \\ u^{(j2)}\big|_{t=0} = 0 \quad \text{on } \Omega^{(j)},\ j = 1, 2, \end{cases} \tag{3.37}$$

together with the boundary conditions:

$$\begin{cases} u^{(12)} - u^{(22)}\big|_{S_{F,T}^{(2)}} = 0, \\ \Pi^{(2)}\big[\nu^{(2)} \mathrm{D}\big(u^{(22)}\big) N^{(2)} - \nu^{(1)} \mathrm{D}\big(u^{(12)}\big) N^{(2)}\big]\big|_{S_{F,T}^{(2)}} = \vec{d}^{(2)}, \\ q^{(12)} - q^{(22)} + 2\nu^{(2)} N^{(2)} \cdot \big(\nabla u^{(22)} N^{(2)}\big) - 2\nu^{(1)} N^{(2)} \cdot \big(\nabla u^{(12)} N^{(2)}\big) \\ \qquad - \dfrac{\sigma^{(2)}}{2} N^{(2)} \cdot \displaystyle\int_0^t \Delta^{(2)}\big(u^{(12)} + u^{(22)}\big)\, d\tau \Big|_{S_{F,T}^{(2)}} \\ \quad = d^{(2)} + \dfrac{\sigma^{(2)}}{2} \displaystyle\int_0^t \big(D^{(2)} + H^{(2)}\big)\, d\tau, \end{cases} \tag{3.38}$$

where $f^{(j)}$, $g^{(j)}$, $\vec{d}^{(2)}$, $d^{(2)}$, $H^{(2)}$ are determined by

$$f^{(j)} = -\nu^{(j)} \sum_{k'''} \bigl[\nabla^2\bigl(\eta_{k'''} u^{(j2)(k''')}\bigr) - \eta_{k'''}\bigl({}^t J_{k'''}^{-1} L_{k'''} \nabla\bigr)^2 u^{(j2)(k''')}\bigr] + \sum_{k'''} \bigl[\nabla\bigl(\eta_{k'''} q^{(j2)(k''')}\bigr) - \eta_{k'''} {}^t L_{k'''} {}^t J_{k'''}^{-1} L_{k'''} \nabla q^{(j2)(k''')}\bigr] + \sum_{k'''} \bigl[\eta_{k'''}\bigl(\rho_0^{(j)}(x) - \rho_0^{(j)}\bigl(\xi^{(k''')}\bigr)\bigr) u_t^{(j2)(k''')}\bigr]$$

and

$$g^{(j)} = \sum_{k'''} \bigl[\nabla \eta_{k'''} \cdot u^{(j2)(k''')} + \eta_{k'''} \nabla \cdot u^{(j2)(k''')}\bigr],$$

$$\vec{d}^{(2)} = \sum_{k'''} \Pi^{(2)} \left\{ \sum_{j=1}^{2} (-1)^j \nu^{(j)} \bigl[\mathrm{D}\bigl(\eta_{k'''} u^{(j2)(k''')}\bigr) N^{(2)} - \eta_{k'''} \Pi_{k'''}\bigl[{}^t L_{k'''} \bar{\mathrm{D}}_x^{(k''')}\bigl(L_{k'''} u^{(j2)(k''')}\bigr) L_{k'''} n_{k'''}\bigr]\bigr] \right\},$$

$$d^{(2)} = 2 \sum_{k'''} \sum_{j=1}^{2} (-1)^j \nu^{(j)} \bigl\{N^{(2)} \cdot \bigl[\nabla\bigl(\eta_{k'''} u^{(j2)(k''')}\bigr) N^{(2)}\bigr] - \eta_{k'''} n_{k'''} \cdot \bigl({}^t L_{k'''} {}^t J_{k'''}^{-1} L_{k'''} \nabla u^{(j2)(k''')} n_{k'''}\bigr)\bigr\},$$

$$H^{(2)} = -\sum_{k'''} \sum_{j=1}^{2} \bigl[N^{(2)} \cdot \Delta^{(2)}\bigl(\eta_{k'''} u^{(j2)(k''')}\bigr) - \eta_{k'''} n_{k'''} \cdot \Delta_{k'''}^{(2)} u^{(j2)(k''')}\bigr],$$

with $\Pi_{k'''} f \stackrel{\mathrm{def}}{=} f - (f \cdot n_{k'''}) n_{k'''}$, $n_{k'''} \stackrel{\mathrm{def}}{=} N^{(2)}(\xi^{(k''')})$, and $\bar{\mathrm{D}}_x^{(k''')}(v) \stackrel{\mathrm{def}}{=} {}^t J_{k'''}^{-1} L_{k'''} \nabla_x v + {}^t({}^t J_{k'''}^{-1} L_{k'''} \nabla_x v)$. Moreover, we can reformulate $g^{(j)}$ as

$$g^{(j)} = \sum_{k'''} \nabla \cdot \bigl(\eta_{k'''}\bigl(I - {}^t L_{k'''} {}^t J_{k'''}^{-1} L_{k'''}\bigr) u^{(j2)(k''')}\bigr) + \sum_{k'''} \bigl(\nabla \cdot \bigl(\eta_{k'''} {}^t L_{k'''} {}^t J_{k'''}^{-1} L_{k'''}\bigr)\bigr) \cdot u^{(j2)(k''')} \stackrel{\mathrm{def}}{=} \nabla \cdot R^{(j)},$$

with

$$R^{(j)} = \sum_{k'''} \eta_{k'''}\bigl(I - {}^t L_{k'''} {}^t J_{k'''}^{-1} L_{k'''}\bigr) u^{(j2)(k''')} - \sum_{k'''} \frac{1}{4\pi} \nabla \int_{\Omega^{(j)}} \frac{1}{|x-y|} \bigl(\nabla \eta_{k'''} \cdot \bigl({}^t L_{k'''} {}^t J_{k'''}^{-1} L_{k'''}\bigr)\bigr) \cdot u^{(j2)(k''')}\, dy - \sum_{k'''} \frac{1}{4\pi} \nabla \int_{\Omega^{(j)}} \frac{1}{|x-y|} \bigl(\eta_{k'''} L_{k'''} \nabla \cdot \bigl({}^t J_{k'''}^{-1} L_{k'''}\bigr)\bigr) \cdot u^{(j2)(k''')}\, dy.$$

Then thanks to Lemma 3.2, (3.35)–(3.36) has a unique solution $(w^{(1)(k''')}, s^{(1)(k''')}, w^{(2)(k''')}, s^{(2)(k''')})$ with $w^{(j)(k''')} \in H_\gamma^{l+2,l/2+1}(\mathbb{D}_T^{(j)})$, $\nabla s^{(j)(k''')} \in H_\gamma^{l,l/2}(\mathbb{D}_T^{(j)})$ for $\gamma \geqslant \gamma_0 \gg 1$, and there holds

$$\sum_{j=1}^{2} \bigl(\bigl\|w^{(j)(k''')}\bigr\|_{H_\gamma^{l+2,l/2+1}(\mathbb{D}_T^{(j)})} + \bigl\|\nabla s^{(j)(k''')}\bigr\|_{H_\gamma^{l,l/2}(\mathbb{D}_T^{(j)})}\bigr)$$

$$\leqslant C\sigma^{(2)} \big\| \big(\zeta_{k'''} D^{(2)}\big)\big(Y_{k'''}^{-1} \circ Z_{k'''}^{-1}(z)\big) \big\|_{H_\gamma^{l-1/2,l/2-1/4}(\mathbb{R}_T^2)},$$

which ensures that

$$\sum_{j=1}^{2} \big(\| u^{(j2)(k''')} \|_{H_\gamma^{l+2,l/2+1}(Q_T^{(j)(k''')})} + \| \nabla q^{(j2)(k''')} \|_{H_\gamma^{l,l/2}(Q_T^{(j)(k''')})} \big) \\ \leqslant C\sigma^{(2)} \| \zeta_{k'''} D^{(2)} \|_{H_\gamma^{l-1/2,l/2-1/4}(S_{F,T}^{(2)})}, \tag{3.39}$$

where $Q_T^{(j)(k''')} \stackrel{\text{def}}{=} \Omega_{k'''}^{(j)} \times (0,T)$.

As in Step 1, we can obtain

$$\sum_{j=1}^{2} \big[\| f^{(j)} \|_{H_\gamma^{l,l/2}(Q_T^{(j)})} + \| g^{(j)} \|_{H_\gamma^{l+1,l/2+1/2}(Q_T^{(j)})} + \| R^{(j)} \|_{H_\gamma^{0,l/2+1}(Q_T^{(j)})} \big] \\ + \big\| \big(\vec{d}^{(2)}, d^{(2)}\big) \big\|_{H_\gamma^{l+1/2,l/2+1/4}(S_{F,T}^{(2)})} + \| H^{(2)} \|_{H_\gamma^{l-1/2,l/2-1/4}(S_{F,T}^{(2)})} \\ \leqslant \mu_7(\lambda,\gamma)\sigma^{(2)} \| D^{(2)} \|_{H_\gamma^{l-1/2,l/2-1/4}(S_{F,T}^{(2)})} + C(\lambda) \sum_{j,k'''} \| q^{(j)(k''')} \|^2_{H_\gamma^{l,l/2}(Q_T^{(k''')})}. \tag{3.40}$$

Thus, it remains to estimate $\|s^{(j)(k''')}\|_{H_\gamma^{0,l/2+1/4}(K_{m\lambda}^{(k''')} \times (0,T))}$, where $K_{m\lambda}^{(1)(k''')} \stackrel{\text{def}}{=} \{z \in \mathbb{R}_+^3 \mid |z| \leqslant m\lambda\}$ and $K_{m\lambda}^{(2)(k''')} \stackrel{\text{def}}{=} \{z \in \mathbb{R}_-^3 \mid |z| \leqslant m\lambda\}$. To handle it, we derive from (3.35)–(3.36) that

$$\begin{cases} \nabla^2 s^{(j)(k''')} = 0 \quad \text{in } \mathbb{D}_T^{(j)}, \\ s^{(1)(k''')} - s^{(2)(k''')}\big|_{z_3=0} = 2\nu^{(1)} \dfrac{\partial w_3^{(1)(k''')}}{\partial z_3} - 2\nu^{(2)} \dfrac{\partial w_3^{(2)(k''')}}{\partial z_3} \\ \quad + \dfrac{\sigma^{(2)}}{2} \displaystyle\int_0^t \big[\nabla'^2 \big(w_3^{(1)(k''')} + w_3^{(2)(k''')}\big) + \big(\zeta_{k'''} D^{(2)}\big)\big(Y_{k'''}^{-1} \circ Z_{k'''}^{-1}(z)\big)\big] \, d\tau \Big|_{z_3=0} \\ \quad \stackrel{\text{def}}{=} A \quad \text{on } \mathbb{R}_T^2, \\ \dfrac{1}{\rho_0^{(1)}(\xi^{(k''')})} \dfrac{\partial s^{(1)(k''')}}{\partial z_3} - \dfrac{1}{\rho_0^{(2)}(\xi^{(k''')})} \dfrac{\partial s^{(2)(k''')}}{\partial z_3} \Big|_{z_3=0} \\ \quad = \dfrac{\nu^{(1)}}{\rho_0^{(1)}(\xi^{(k''')})} \nabla^2 w_3^{(1)(k''')} - \dfrac{\nu^{(2)}}{\rho_0^{(2)}(\xi^{(k''')})} \nabla^2 w_3^{(2)(k''')} \Big|_{z_3=0} \stackrel{\text{def}}{=} B \quad \text{on } \mathbb{R}_T^2. \end{cases} \tag{3.41}$$

Extending the functions appeared in (3.41) from $(0,T)$ to $(0,+\infty)$ with preservation of their classes, and then applying the partial Fourier–Laplace transformation

$$\hat{f}\big(\xi', z_3, \zeta\big) \stackrel{\text{def}}{=} \int_0^\infty e^{-\zeta t} \, dt \int_{\mathbb{R}^2} e^{-i\xi' \cdot z'} f\big(z', z_3, t\big) \, dz', \quad \zeta = \gamma + i\xi_0,$$

to (3.41), we get

$$\begin{cases} \left(\xi'^2 - \dfrac{d^2}{dz_3^2}\right)\hat{s}^{(1)(k''')} = 0, \quad \left(\xi'^2 - \dfrac{d^2}{dz_3^2}\right)\hat{s}^{(2)(k''')} = 0, \\ \hat{s}^{(1)(k''')} - \hat{s}^{(2)(k''')}\big|_{z_3=0} = \hat{A}, \\ \dfrac{1}{\rho_0^{(1)}(\xi^{(k''')})}\dfrac{d\hat{s}^{(1)(k''')}}{dz_3} - \dfrac{1}{\rho_0^{(2)}(\xi^{(k''')})}\dfrac{d\hat{s}^{(2)(k''')}}{dz_3}\Big|_{z_3=0} = \hat{B}. \end{cases}$$

Then under the assumption that the solution decays to 0 as $|z_3|$ tends to $+\infty$, we obtain

$$\begin{aligned} \hat{s}^{(1)(k''')}\big(\xi', z_3, \zeta\big) &= \frac{\rho_0^{(1)}(\xi^{(k''')})}{\rho_0^{(1)}(\xi^{(k''')}) + \rho_0^{(2)}(\xi^{(k''')})}\hat{A}e^{-|\xi'|z_3} \\ &\quad - \frac{\rho_0^{(1)}(\xi^{(k''')})\rho_0^{(2)}(\xi^{(k''')})}{\rho_0^{(1)}(\xi^{(k''')}) + \rho_0^{(2)}(\xi^{(k''')})}\frac{\hat{B}}{|\xi'|}e^{-|\xi'|z_3}, \quad z_3 > 0, \\ \hat{s}^{(2)(k''')}\big(\xi', z_3, \zeta\big) &= -\frac{\rho_0^{(2)}(\xi^{(k''')})}{\rho_0^{(1)}(\xi^{(k''')}) + \rho_0^{(2)}(\xi^{(k''')})}\hat{A}e^{|\xi'|z_3} \\ &\quad - \frac{\rho_0^{(1)}(\xi^{(k''')})\rho_0^{(2)}(\xi^{(k''')})}{\rho_0^{(1)}(\xi^{(k''')}) + \rho_0^{(2)}(\xi^{(k''')})}\frac{\hat{B}}{|\xi'|}e^{|\xi'|z_3}, \quad z_3 < 0. \end{aligned} \tag{3.42}$$

While note that [18]

$$\begin{aligned} \mathscr{F}_{z'\mapsto\xi'}\left(\frac{1}{2\pi}\frac{z_3}{|z|^3}\right)\big(\xi', z_3\big) &= e^{-|\xi'|z_3}, \quad z_3 > 0, \\ \mathscr{F}_{z'\mapsto\xi'}\left(\frac{1}{2\pi}\frac{1}{|z|}\right)\big(\xi', z_3\big) &= \frac{e^{-|\xi'|z_3}}{|\xi'|}, \quad z_3 > 0, \end{aligned}$$

and let $\tilde{f}(z,\zeta) \stackrel{\text{def}}{=} \int_0^\infty e^{-\zeta t} f(z,t)\,dt$ be the Laplace transformation of f, we get by taking inverse Fourier transform to (3.42) that

$$\begin{aligned} \tilde{s}^{(1)(k''')}(z,\zeta) &= -\frac{1}{2\pi}\frac{\rho_0^{(1)}(\xi^{(k''')})\rho_0^{(2)}(\xi^{(k''')})}{\rho_0^{(1)}(\xi^{(k''')}) + \rho_0^{(2)}(\xi^{(k''')})}\int_{\mathbb{R}^2}\left(\frac{1}{\rho_0^{(2)}(\xi^{(k''')})}\right. \\ &\quad \left. \times \frac{\partial}{\partial z_3}\frac{1}{|z-(y',0)|}\tilde{A}\big(y',\zeta\big) + \frac{1}{|z-(y',0)|}\tilde{B}\big(y',\zeta\big)\right)dy', \quad z_3 > 0, \\ \tilde{s}^{(2)(k''')}(z,\zeta) &= -\frac{1}{2\pi}\frac{\rho_0^{(1)}(\xi^{(k''')})\rho_0^{(2)}(\xi^{(k''')})}{\rho_0^{(1)}(\xi^{(k''')}) + \rho_0^{(2)}(\xi^{(k''')})}\int_{\mathbb{R}^2}\left(\frac{1}{\rho_0^{(1)}(\xi^{(k''')})}\right. \\ &\quad \left. \times \frac{\partial}{\partial z_3}\frac{1}{|z-(y',0)|}\tilde{A}\big(y',\zeta\big) + \frac{1}{|z-(y',0)|}\tilde{B}\big(y',\zeta\big)\right)dy', \quad z_3 < 0, \end{aligned}$$

from which, we can get by the potential theory [18] that

$$\sum_{j=1}^{2}\left\|s^{(j)(k''')}\right\|^2_{H_\gamma^{0,l/2+1/4}(K_{m\lambda}^{(j)(k''')}\times(0,T))}$$
$$\leqslant C\left(\lambda+\gamma^{-1}\right)\sum_{j=1}^{2}\left\|w^{(j)(k''')}\right\|^2_{H_\gamma^{l+2,l/2+1}(\mathbb{D}_T^{(j)})}$$
$$+\sigma^{(2)2}\gamma^{-1}\left\|\left(\zeta_{k'''}D^{(2)}\right)\left(Y_{k'''}^{-1}\circ Z_{k'''}^{-1}(z)\right)\right\|^2_{H_\gamma^{0,l/2-1/4}(\mathbb{R}_T^2)}. \tag{3.43}$$

Notice that

$$\sum_{j=1}^{2}\left\|s^{(j)(k''')}\right\|_{H_\gamma^{l,l/2}(K_{m\lambda}^{(j)(k''')}\times(0,T))}\leqslant\epsilon\sum_{j=1}^{2}\left\|\nabla s^{(j)(k''')}\right\|_{H_\gamma^{l,0}(K_{m\lambda}^{(j)(k''')}\times(0,T))}$$
$$+\left(c(\epsilon)\gamma^{-l/2-1/4}+C\gamma^{-1/4}\right)\sum_{j=1}^{2}\left\|s^{(j)(k''')}\right\|_{H_\gamma^{0,l/2+1/4}(K_{m\lambda}^{(j)(k''')}\times(0,T))},$$

for any $\epsilon>0$, we deduce from (3.40) and (3.43) that

$$\sum_{j=1}^{2}\left[\left\|f^{(j)}\right\|_{H_\gamma^{l,l/2}(Q_T^{(j)})}+\left\|g^{(j)}\right\|_{H_\gamma^{l+1,l/2+1/2}(Q_T^{(j)})}+\left\|R^{(j)}\right\|_{H_\gamma^{0,l/2+1}(Q_T^{(j)})}\right]$$
$$+\left\|\left(\vec{d}^{(2)},d^{(2)}\right)\right\|_{H_\gamma^{l+1/2,l/2+1/4}(S_{F,T}^{(2)})}+\left\|H^{(2)}\right\|_{H_\gamma^{l-1/2,l/2-1/4}(S_{F,T}^{(2)})}$$
$$\leqslant\mu_8(\lambda,\gamma)\sigma^{(2)}\left\|D^{(2)}\right\|_{H_\gamma^{l-1/2,l/2-1/4}(S_{F,T}^{(2)})}. \tag{3.44}$$

Step 3. Similar to Step 1 and Step 2, we define $\mathscr{L}_3F\overset{\text{def}}{=}(u^{(13)},q^{(13)},u^{(23)},q^{(23)})$ with $(u^{(13)},q^{(13)},u^{(23)},q^{(23)})$ satisfying

$$\begin{cases}\rho_0^{(1)}u_t^{(13)}-\nu^{(1)}\nabla^2u^{(13)}+\nabla q^{(13)}=-\bar{f}^{(1)}-f^{(1)}\quad\text{in }Q_T^{(1)},\\ \rho_0^{(2)}u_t^{(23)}-\nu^{(2)}\nabla^2u^{(23)}+\nabla q^{(23)}=-f^{(2)}\quad\text{in }Q_T^{(2)},\\ \nabla\cdot u^{(13)}=-\bar{g}^{(1)}-g^{(1)}\quad\text{in }Q_T^{(1)},\quad\nabla\cdot u^{(23)}=-g^{(2)}\quad\text{in }Q_T^{(2)},\\ u^{(13)}\big|_{t=0}=0\quad\text{on }\Omega^{(1)},\quad u^{(23)}\big|_{t=0}=0\quad\text{on }\Omega^{(2)},\end{cases} \tag{3.45}$$

together with the boundary conditions

$$\begin{cases}u^{(13)}-u^{(23)}\big|_{S_{F,T}^{(2)}}=0,\quad u^{(23)}\big|_{S_{B,T}}=0,\\ \nu^{(1)}\Pi^{(1)}\left[\mathrm{D}\left(u^{(13)}\right)N^{(1)}\right]\big|_{S_{F,T}^{(1)}}=-\vec{d}^{(1)},\\ -q^{(13)}+2\nu^{(1)}N^{(1)}\cdot\left(\nabla u^{(13)}N^{(1)}\right)\big|_{S_{F,T}^{(1)}}=-d^{(1)},\\ \Pi^{(2)}\left[\nu^{(2)}\mathrm{D}\left(u^{(23)}N^{(2)}\right)-\nu^{(1)}\mathrm{D}\left(u^{(13)}\right)N^{(2)}\right]\big|_{S_{F,T}^{(2)}}=-\vec{d}^{(2)},\\ q^{(13)}-q^{(23)}+2\nu^{(2)}N^{(2)}\cdot\left(\nabla u^{(23)}N^{(2)}\right)-2\nu^{(1)}N^{(2)}\cdot\left(\nabla u^{(13)}N^{(2)}\right)\big|_{S_{F,T}^{(2)}}=-d^{(2)}.\end{cases} \tag{3.46}$$

Thanks to Theorem 3.2, (3.45)–(3.46) has a unique solution $(u^{(j3)}, q^{(j3)}) \in H_\gamma^{l+2,l/2+1}(Q_T^{(j)}) \times H_\gamma^{l+1,1,l/2}(Q_T^{(j)})$, $j = 1, 2$ for $\gamma \geqslant \gamma_0 \gg 1$. Furthermore, thanks to (3.33) and (3.44), we have

$$\sum_{j=1}^{2}\big(\|u^{(j3)}\|_{H_\gamma^{l+2,l/2+1}(Q_T^{(j)})} + \|q^{(j3)}\|_{H_\gamma^{l+l,1,l/2}(Q_T^{(j)})}\big) \leqslant \mu_9(\lambda,\gamma)\sum_{j=1}^{2}\sigma^{(j)}\|D^{(j)}\|_{H_\gamma^{l-1/2,l/2-1/4}(S_{F,T}^{(j)})}. \tag{3.47}$$

Now let $(u^{(1)}, q^{(1)}, u^{(2)}, q^{(2)}) \stackrel{\text{def}}{=} \sum_{j=1}^{3}(u^{(1j)}, q^{(1j)}, u^{(2j)}, q^{(2j)})$ satisfy (3.7)–(3.8) with $(D^{(1)}, D^{(2)})$ there being replaced by $(D^{(1)} + \mathscr{M}_1 D, D^{(2)} + \mathscr{M}_2 D)$ where

$$\begin{cases} \mathscr{M}_1 D \stackrel{\text{def}}{=} H^{(1)} - N^{(1)} \cdot \Delta^{(1)} u^{(13)}, \\ \mathscr{M}_2 D \stackrel{\text{def}}{=} H^{(2)} - N^{(2)} \cdot \Delta^{(2)}\big(u^{(13)} + u^{(23)}\big). \end{cases} \tag{3.48}$$

However, thanks to (3.33), (3.44) and (3.47), we get

$$\sum_{j=1}^{2}\|\mathscr{M}_j D\|_{H_\gamma^{l-1/2,l/2-1/4}(S_{F,T}^{(j)})} \leqslant \mu_{10}(\lambda,\gamma)\sum_{j=1}^{2}\sigma^{(j)}\|D^{(j)}\|_{H_\gamma^{l-1/2,l/2-1/4}(S_{F,T}^{(j)})}. \tag{3.49}$$

Taking $\lambda \ll 1$ and then $\gamma \gg 1$ such that $\mu_{10}(\lambda,\gamma)\max_{i=1,2}\sigma^{(i)} \leqslant 1/2 < 1$, then, $\mathscr{M} = (\mathscr{M}_1, \mathscr{M}_2)$ is a contraction. Therefore, thanks to the argument at the beginning of this subsection, (3.7)–(3.8) is solvable for $\gamma \geqslant \gamma_0 \gg 1$ and its solution $(u''^{(1)}, q''^{(1)}, u''^{(2)}, q''^{(2)})$ satisfies

$$\sum_{j=1}^{2}\big(\|u''^{(j)}\|_{H_\gamma^{l+2,l/2+1}(Q_T^{(j)})} + \|q''^{(j)}\|_{H_\gamma^{l+l,1,l/2}(Q_T^{(j)})}\big) \leqslant C\sum_{j=1}^{2}\sigma^{(j)}\|D^{(j)}\|_{H_\gamma^{l-1/2,l/2-1/4}(S_{F,T}^{(j)})}. \tag{3.50}$$

This completes the proof of Theorem 3.3. □

Combining Theorem 3.2 with Theorem 3.3, we obtain

Proposition 3.3. *Under the assumptions of Theorem* 3.2 *and Theorem* 3.3, (3.1)–(3.2) *with* $v_0^{(j)} = 0$, $j = 1, 2$, *has a unique solution* $(u^{(j)}, q^{(j)}) \in H_\gamma^{l+2,l/2+1}(Q_T^{(j)}) \times H_\gamma^{l+1,1,l/2}(Q_T^{(j)})$ *for* $\gamma \geqslant \gamma_0 \gg 1$. *Moreover, there holds*

$$\begin{aligned}
&\sum_{j=1}^{2}\big(\|u^{(j)}\|_{H_\gamma^{l+2,l/2+1}(Q_T^{(j)})} + \|q^{(j)}\|_{H_\gamma^{l+1,1,l/2}(Q_T^{(j)})}\big) \\
&\quad \leqslant C\sum_{j=1}^{2}\big[\|f^{(j)}\|_{H_\gamma^{l,l/2}(Q_T^{(j)})} + \|g^{(j)}\|_{H_\gamma^{l+1,0}(Q_T^{(j)})} + \|R^{(j)}\|_{H_\gamma^{0,l/2+1}(Q_T^{(j)})} \\
&\qquad + \|\vec{b}^{(j)}\|_{H_\gamma^{l+1/2,l/2+1/4}(S_{F,T}^{(j)})} + \|\bar{b}^{(j)}\|_{H_\gamma^{l+1/2,1/2,l/2}(S_{F,T}^{(j)})} + \sigma^{(j)}\|B^{(j)}\|_{H_\gamma^{l-1/2,l/2-1/4}(S_{F,T}^{(j)})}\big].
\end{aligned} \tag{3.51}$$

3.3. Proof of Theorem 3.1

The main aim of this subsection is to prove Theorem 3.1.

Proof of Theorem 3.1. The main idea of the proof is motivated by [15,21]. Firstly thanks to Lemma 6.2 of [15], there exists $(w^{(1)}, w^{(2)}) \in W_2^{l+2,l/2+1}(Q_\infty^{(1)}) \times W_2^{l+2,l/2+1}(Q_\infty^{(2)})$ such that $w^{(1)}|_{t=0} = v_0^{(1)}$, $w^{(2)}|_{t=0} = v_0^{(2)}$, and

$$\|w^{(j)}\|_{W_2^{l+2,l/2+1}(Q_\infty^{(j)})} \leqslant C\|v_0^{(j)}\|_{W_2^{l+1}(\Omega^{(j)})}, \quad j=1,2.$$

We further construct $w'^{(j)} \in W_2^{l+2,l/2+1}(Q_\infty^{(j)})$, $j=1,2$, such that

$$\begin{gathered}
w'^{(j)}\big|_{t=0} = 0, \qquad w'^{(1)} - w'^{(2)}\big|_{S_F^{(2)}} = w^{(2)} - w^{(1)}\big|_{S_F^{(2)}}, \qquad w'^{(2)}\big|_{S_B} = -w^{(2)}\big|_{S_B}, \\
\sum_{j=1}^{2}\|w'^{(j)}\|_{W_2^{l+2,l/2+1}(Q_\infty^{(j)})} \leqslant C\sum_{j=1}^{2}\|w^{(j)}\|_{W_2^{l+2,l/2+1}(Q_\infty^{(j)})} \leqslant C\sum_{j=1}^{2}\|v_0^{(j)}\|_{W_2^{l+1}(\Omega^{(j)})}.
\end{gathered} \tag{3.52}$$

Let $u'^{(j)} \stackrel{\text{def}}{=} u^{(j)} - w^{(j)} - w'^{(j)}$, $j=1,2$. Then $(u'^{(1)}, q^{(1)}, u'^{(2)}, q^{(2)})$ satisfies (3.1)–(3.2) with $(f^{(j)}, g^{(j)}, \vec{b}^{(j)}, \bar{b}^{(j)}, B^{(j)}, v_0^{(j)})$ there being replaced by $(f'^{(j)}, g'^{(j)}, \vec{b}'^{(j)}, \bar{b}'^{(j)}, B'^{(j)}, 0)$ and

$$\begin{aligned}
f'^{(j)} &= f^{(j)} - w_t^{(j)} - w_t'^{(j)} + \frac{\nu^{(j)}}{\rho_0^{(j)}}\big(\nabla^2 w^{(j)} + \nabla^2 w'^{(j)}\big), \\
g'^{(j)} &= g^{(j)} - \nabla\cdot\big(w^{(j)} + w'^{(j)}\big), \\
\vec{b}'^{(1)} &= \vec{b}^{(1)} - \nu^{(1)}\Pi^{(1)}\big[\mathrm{D}\big(w^{(1)} + w'^{(1)}\big)N^{(1)}\big], \\
\bar{b}'^{(1)} &= \bar{b}^{(1)} - 2\nu^{(1)}N^{(1)}\cdot\big(\nabla w^{(1)}N^{(1)} + \nabla w'^{(1)}N^{(1)}\big), \\
B'^{(1)} &= B^{(1)} + N^{(1)}\cdot\Delta^{(1)}\big(w^{(1)} + w'^{(1)}\big), \\
\vec{b}'^{(2)} &= \vec{b}^{(2)} - \Pi^{(2)}\left[\sum_{j=1}^{2}(-1)^j\nu^{(j)}\mathrm{D}\big(w^{(j)} + w'^{(j)}\big)N^{(2)}\right], \\
\bar{b}'^{(2)} &= \bar{b}^{(2)} - 2\sum_{j=1}^{2}(-1)^j\nu^{(j)}N^{(2)}\cdot\big(\nabla w^{(j)}N^{(2)} + \nabla w'^{(j)}N^{(2)}\big), \\
B'^{(2)} &= B^{(2)} + N^{(2)}\cdot\Delta^{(2)}\left[\sum_{j=1}^{2}\big(w^{(j)} + w'^{(j)}\big)\right].
\end{aligned}$$

However, by the virtue of Lemma 6.3 of [15], we have

$$T^{-l}\|f\|^2_{L^2(Q_T)} \leqslant C\int_0^\infty\int_0^\infty \|f_0(\cdot, t-\tau) - f_0(\cdot, t)\|^2_{L^2(\Omega)}\frac{d\tau\,dt}{\tau^{l+1}} \leqslant C\|f\|^2_{W_2^{0,l/2}(Q_\infty)},$$

and

$$\sum_{j=1}^{2}\left(\|w^{(j)}\|^{(l+2)}_{Q_T^{(j)}}+\|w'^{(j)}\|^{(l+2)}_{Q_T^{(j)}}\right)\leqslant C\sum_{j=1}^{2}\|v_0^{(j)}\|_{W_2^{l+1}(\Omega^{(j)})},\tag{3.53}$$

which implies

$$\sum_{j=1}^{2}\|f'^{(j)}\|^{(l)}_{Q_T^{(j)}}\leqslant C\sum_{j=1}^{2}\left(\|f^{(j)}\|^{(l)}_{Q_T^{(j)}}+\|v_0^{(j)}\|_{W_2^{l+1}(\Omega^{(j)})}\right).$$

While applying Corollary 6.3 of [15], we obtain

$$\|f'^{(j)}\|^2_{H_0^{l,l/2}(Q_T^{(j)})}\leqslant C\left(\|f'^{(j)}\|^2_{W_2^{l,l/2}(Q_T^{(j)})}+T^{-l}\|f'^{(j)}\|^2_{L^2(Q_T^{(j)})}\right)\leqslant C\|f'^{(j)}\|^{(l)2}_{Q_T^{(j)}},$$

whence

$$\sum_{j=1}^{2}\|f'^{(j)}\|_{H_0^{l,l/2}(Q_T^{(j)})}\leqslant C\sum_{j=1}^{2}\left(\|f^{(j)}\|^{(l)}_{Q_T^{(j)}}+\|v_0^{(j)}\|_{W_2^{l+1}(\Omega^{(j)})}\right).\tag{3.54}$$

It is easy to see that

$$\sum_{j=1}^{2}\|g'^{(j)}\|_{H_0^{l+1,0}(Q_T^{(j)})}\leqslant C\sum_{j=1}^{2}\left(\|g^{(j)}\|_{W_2^{l+1,0}(Q_T^{(j)})}+\|v_0^{(j)}\|_{W_2^{l+1}(\Omega^{(j)})}\right).\tag{3.55}$$

We rewrite $g'^{(j)}$ as $g'^{(j)}=\nabla\cdot R_1^{(j)}$ with $R_1^{(j)}=R^{(j)}-w^{(j)}-w'^{(j)}$ and

$$\|R_1^{(j)}\|_{W_2^{0,l/2+1}(Q_T^{(j)})}\leqslant C\left(\|R^{(j)}\|_{W_2^{0,l/2+1}(Q_T^{(j)})}+\|v_0^{(j)}\|_{W_2^{l+1}(\Omega^{(j)})}\right).$$

Let $R_2^{(j)}=\nabla\Phi^{(j)}$ where $\Phi^{(j)}$ is the solution of

$$\nabla^2\Phi^{(j)}=g'^{(j)}\quad\text{in }\Omega^{(j)},\qquad \Phi^{(j)}\big|_{\partial\Omega^{(j)}}=0.$$

Since $g'^{(j)}|_{t=0}=g^{(j)}|_{t=0}-\nabla\cdot v_0^{(j)}=0$, we get

$$g'^{(j)}=\nabla\cdot R_2^{(j)},\qquad R_2^{(j)}\big|_{t=0}=0.$$

Thus we obtain

$$\begin{aligned}\|R_2^{(j)}\|_{H_0^{0,l/2+1}(Q_T^{(j)})}&=\|\nabla\Phi_t^{(j)}\|_{H_0^{0,l/2}(Q_T^{(j)})}\leqslant C\|R_{1t}^{(j)}\|_{H_0^{0,l/2}(Q_T^{(j)})}\\&\leqslant C\left(\|R_t^{(j)}\|_{L^2(\Omega^{(j)};\dot W_2^{l/2}(0,T))}+T^{-l/2}\|R_t^{(j)}\|_{L^2(Q_T^{(j)})}+\|v_0^{(j)}\|_{W_2^{l+1}(\Omega^{(j)})}\right).\end{aligned}\tag{3.56}$$

Due to the assumption of the theorem, we have

$$\vec b'^{(j)}\big|_{t=0}=0$$

which together with Lemmas 6.1 and 6.3 in [15] gives

$$\sum_{j=1}^{2}\big\|\vec{b}'^{(j)}\big\|_{H_0^{l+1/2,l/2+1/4}(S_{F,T}^{(j)})}\leqslant C\sum_{j=1}^{2}\big(\big\|\vec{b}^{(j)}\big\|_{W_2^{l+1/2,l/2+1/4}(S_{F,T}^{(j)})}+\big\|v_0^{(j)}\big\|_{W_2^{l+1}(\Omega^{(j)})}\big). \tag{3.57}$$

Similarly, we can obtain

$$\sum_{j=1}^{2}\big\|B'^{(j)}\big\|^2_{H_0^{l-1/2,l/2-1/4}(S_{F,T}^{(j)})}\leqslant C\sum_{j=1}^{2}\Bigg\{\big(\big\|B^{(j)}\big\|^{(l-1/2)}_{S_{F,T}^{(j)}}\big)^2+\sum_{|\alpha|=2}\big\|D^\alpha\big(w^{(j)}+w'^{(j)}\big)\big\|^2_{W_2^{l,l/2}(Q_T^{(j)})}$$
$$+T^{-l+1/2}\int_0^T\sum_{|\alpha|=2}\big\|D^\alpha\big(w^{(j)}+w'^{(j)}\big)\big\|^2_{L^2(S_F^{(j)})}\,dt\Bigg\}.$$

Using the interpolation inequality

$$\|f\|^2_{L^2(S_F)}\leqslant\epsilon\|f\|^2_{W_2^l(\Omega)}+C\epsilon^{-\frac{1}{2l-1}}\|f\|^2_{L_2(\Omega)},$$

with $\epsilon=T^{l-1/2}$, we obtain

$$T^{-l+1/2}\int_0^T\|f\|^2_{L^2(S_F)}\,dt\leqslant\int_0^T\|f\|^2_{W_2^l(\Omega)}\,dt+CT^{-l}\int_0^T\|f\|^2_{L_2(\Omega)}\,dt.$$

Thanks to (3.53), we finally arrive at

$$\sum_{j=1}^{2}\big\|B'^{(j)}\big\|_{H_0^{l-1/2,l/2-1/4}(S_{F,T}^{(j)})}\leqslant C\sum_{j=1}^{2}\big(\big\|B^{(j)}\big\|^{(l-1/2)}_{S_{F,T}^{(j)}}+\big\|v_0^{(j)}\big\|_{W_2^{l+1}(\Omega^{(j)})}\big). \tag{3.58}$$

Similarly, we can get

$$\sum_{j=1}^{2}\big\|\bar{b}'^{(j)}\big\|_{H_0^{l+1/2,1/2,l/2}(S_{F,T}^{(j)})}\leqslant C\big(1+T^{1/4}\big)\sum_{j=1}^{2}\big(\big\|\bar{b}^{(j)}\big\|_{W_2^{l+1/2,l/2+1/4}(S_{F,T}^{(j)})}$$
$$+\big\|v_0^{(j)}\big\|_{W_2^{l+1}(\Omega^{(j)})}\big)+CT^{-l/2}\big\|\bar{b}^{(j)}\big\|_{W_2^{1/2,0}(S_{F,T}^{(j)})}. \tag{3.59}$$

From the definition of the space $H_\gamma^{r,r/2}(Q_T)$ $(T<+\infty)$, we know that the spaces $H_\gamma^{r,r/2}(Q_T)$ and $H_0^{r,r/2}(Q_T)$ are equivalent. Thus, (3.1)–(3.2) with $(f^{(j)},g^{(j)},\vec{b}^{(j)},\bar{b}^{(j)},B^{(j)},v_0^{(j)})$ there being replaced by $(f'^{(j)},g'^{(j)},\vec{b}'^{(j)},\bar{b}'^{(j)},B'^{(j)},0)$ satisfies the conditions of Proposition 3.3, then there exists a unique solution $(u'^{(1)},q^{(1)},u'^{(2)},q^{(2)})$ with $(u'^{(j)},q^{(j)})\in H_\gamma^{l+2,l/2+1}(Q_T^{(j)})\times H_\gamma^{l+1,1,l/2}(Q_T^{(j)})$ for $\gamma\geqslant\gamma_0\gg1$ and thanks to (3.54)–(3.59), we have

$$\sum_{j=1}^{2}\big(\big\|u'^{(j)}\big\|^{(l+2)}_{Q_T^{(j)}}+\big\|q^{(j)}\big\|^{(l)}_{Q_T^{(j)}}+\big\|\nabla q^{(j)}\big\|^{(l)}_{Q_T^{(j)}}\big)$$
$$\leqslant C\sum_{j=1}^{2}\big(\big\|u'^{(j)}\big\|_{H_0^{l+2,l/2+1}(Q_T^{(j)})}+\big\|q^{(j)}\big\|_{H_0^{l+1,1,l/2}(Q_T^{(j)})}\big)$$

$$\begin{aligned}
&\leqslant Ce^{\gamma T}\sum_{j=1}^{2}\left(\left\|u'^{(j)}\right\|_{H_\gamma^{l+2,l/2+1}(Q_T^{(j)})}+\left\|q^{(j)}\right\|_{H_\gamma^{l+1,1,l/2}(Q_T^{(j)})}\right)\\
&\leqslant Ce^{\gamma T}\sum_{j=1}^{2}\Big[\left\|f'^{(j)}\right\|_{H_\gamma^{l,l/2}(Q_T^{(j)})}+\left\|g'^{(j)}\right\|_{H_\gamma^{l+1,0}(Q_T^{(j)})}+\left\|R_2^{(j)}\right\|_{H_\gamma^{0,l/2+1}(Q_T^{(j)})}\\
&\quad+\left\|\vec{b}'^{(j)}\right\|_{H_\gamma^{l+1/2,l/2+1/4}(S_{F,T}^{(j)})}+\left\|\bar{b}'^{(j)}\right\|_{H_\gamma^{l+1/2,1/2,l/2}(S_{F,T}^{(j)})}\\
&\quad+\sigma^{(j)}\left\|B'^{(j)}\right\|_{H_\gamma^{l-1/2,l/2-1/4}(S_{F,T}^{(j)})}\Big]\leqslant C(T)\varXi,
\end{aligned}$$

where $C(T)$ is an increasing function of T. From the boundary conditions, we can easily obtain

$$\left\|q^{(1)}\right\|_{W_2^{l+1/2,l/2+1/4}(S_{F,T}^{(1)})}+\left\|q^{(1)}-q^{(2)}\right\|_{W_2^{l+1/2,l/2+1/4}(S_{F,T}^{(2)})}\leqslant C(T)\varXi.$$

Therefore, we obtain the inequality (3.3). In order to get the inequality (3.4), we use the Hölder inequality to get

$$\left\|R_t^{(j)}\right\|_{L^2(Q_T^{(j)})}\leqslant C\left\|R_t^{(j)}\right\|_{L^p(0,T;L^2(\Omega^{(j)}))}T^{1/2-1/p}\leqslant CT^{l/2}\left\|R_t^{(j)}\right\|_{W_2^{0,l/2}(Q_T^{(j)})},$$

that is,

$$T^{-l/2}\left\|R_t^{(j)}\right\|_{L^2(Q_T^{(j)})}\leqslant C\left\|R^{(j)}\right\|_{W_2^{0,l/2+1}(Q_T^{(j)})},$$

where $\frac{1}{p}=\frac{1}{2}-\frac{l}{2}$. Similarly, we have

$$\begin{aligned}
T^{-l/2}\left\|\bar{b}^{(j)}\right\|_{W_2^{1/2,0}(S_{F,T}^{(j)})}&\leqslant C\left\|\bar{b}^{(j)}\right\|_{W_2^{l+1/2,l/2+1/4}(S_{F,T}^{(j)})},\\
T^{-l/2+1/4}\left\|B^{(j)}\right\|_{L^2(S_{F,T}^{(j)})}&\leqslant C\left\|B^{(j)}\right\|_{W_2^{0,l/2-1/4}(S_{F,T}^{(j)})}.
\end{aligned}$$

Then, we obtain the inequality (3.4) from (3.54)–(3.59), and thus Theorem 3.1 is proved. □

3.4. Proof of Theorem 3.2

Since the proof of Theorem 3.2 is similar to that of Theorem 3.3, we will only sketch a proof. Let us consider the elliptic problem

$$\begin{cases}\nabla^2\varphi^{(j)}=g^{(j)}\quad\text{in }Q_T^{(j)},\ j=1,2,\\[4pt] \varphi^{(1)}-\varphi^{(2)}\big|_{S_F^{(2)}}=0,\quad \left.\dfrac{\partial\varphi^{(1)}}{\partial N^{(2)}}-\dfrac{\partial\varphi^{(2)}}{\partial N^{(2)}}\right|_{S_F^{(2)}}=0,\quad \left.\dfrac{\partial\varphi^{(1)}}{\partial N^{(1)}}\right|_{S_F^{(1)}}=0,\quad \left.\dfrac{\partial\varphi^{(2)}}{\partial N_B}\right|_{S_B}=0,\end{cases}$$

from which and the elliptic estimates, we can obtain (see also [21])

$$\sum_{j=1}^{2}\left\|\varphi^{(j)}\right\|_{H_\gamma^{l+3,l/2+3/2}(Q_T^{(j)})}\leqslant C\sum_{j=1}^{2}\left(\left\|g^{(j)}\right\|_{H_\gamma^{l+1,0}(Q_T^{(j)})}+\left\|R^{(j)}\right\|_{H_\gamma^{0,l/2+1}(Q_T^{(j)})}\right).\tag{3.60}$$

Next, we choose divergence free vector fields $(w^{(1)},w^{(2)})$ such that

$$\begin{cases} w^{(1)} - w^{(2)}\big|_{S_F^{(2)}} = \nabla\varphi^{(2)} - \nabla\varphi^{(1)}\big|_{S_F^{(2)}}, \\ w^{(1)}\big|_{S_F^{(1)}} = -\nabla\varphi^{(1)}\big|_{S_F^{(1)}}, \quad w^{(2)}\big|_{S_B} = -\nabla\varphi^{(2)}\big|_{S_B}, \end{cases}$$

and

$$\sum_{j=1}^{2} \big\| w^{(j)} \big\|_{H_\gamma^{l+2,l/2+1}(Q_T^{(j)})} \leqslant C \sum_{j=1}^{2} \big\| \varphi^{(j)} \big\|_{H_\gamma^{l+3,l/2+3/2}(Q_T^{(j)})}. \tag{3.61}$$

Set $u^{(j)} \stackrel{\text{def}}{=} u'^{(j)} - w^{(j)} - \nabla\varphi^{(j)}$, then $u^{(j)}$ satisfies

$$\rho_0^{(j)} u_t^{(j)} - \nu^{(j)}\nabla^2 u^{(j)} + \nabla q'^{(j)} = \rho_0^{(j)} f'^{(j)} \stackrel{\text{def}}{=} \bar{f}^{(j)} \quad \text{in } Q_T^{(j)},$$

with

$$\begin{aligned} \bar{f}^{(j)} &= \rho_0^{(j)} f^{(j)} - \rho_0^{(j)}\big(\nabla\varphi_t^{(j)} + w_t^{(j)}\big) + \nu^{(j)}\nabla^2\big(w^{(j)} + \nabla\varphi^{(j)}\big) \\ &\stackrel{\text{def}}{=} \Phi^{(j)} + \nabla\Psi^{(j)} \in H_\gamma^{l,l/2}\big(Q_T^{(j)}\big), \end{aligned}$$

where $\Phi^{(j)}, \nabla\Psi^{(j)} \in H_\gamma^{l,l/2}(Q_T^{(j)})$, $\operatorname{div}\Phi^{(j)} = 0$ and $\nabla\Psi^{(j)} \in L^2(Q_T^{(j)})$ with $\Psi^{(j)}|_{\partial\Omega^{(j)}} = 0$ (see [6], Hodge decomposition). Therefore, we have

$$C^{-1}\big\|\bar{f}^{(j)}\big\|_{H_\gamma^{l,l/2}(Q_T^{(j)})} \leqslant \big\|\Phi^{(j)}\big\|_{H_\gamma^{l,l/2}(Q_T^{(j)})} + \big\|\nabla\Psi^{(j)}\big\|_{H_\gamma^{l,l/2}(Q_T^{(j)})} \leqslant C\big\|\bar{f}^{(j)}\big\|_{H_\gamma^{l,l/2}(Q_T^{(j)})}. \tag{3.62}$$

Set $q^{(j)} \stackrel{\text{def}}{=} q'^{(j)} - \Psi^{(j)}$, then $(u^{(j)}, q^{(j)})$ satisfies

$$\begin{cases} \rho_0^{(j)} u_t^{(j)} - \nu^{(j)}\nabla^2 u^{(j)} + \nabla q^{(j)} = \Phi^{(j)} \quad \text{in } Q_T^{(j)}, \\ \nabla\cdot u^{(j)} = 0, \quad u^{(j)}\big|_{t=0} = 0, \quad j = 1, 2, \end{cases} \tag{3.63}$$

together with the boundary conditions

$$\begin{cases} \Pi^{(1)}\big[\nu^{(1)}\mathrm{D}\big(u^{(1)}\big)N^{(1)}\big]\big|_{S_{F,T}^{(1)}} = \vec{b}^{(1)} - \Pi^{(1)}\big[\nu^{(1)}\mathrm{D}\big(\nabla\varphi^{(1)} + w^{(1)}\big)N^{(1)}\big]\big|_{S_{F,T}^{(1)}} \stackrel{\text{def}}{=} \vec{d}^{(1)}, \\ -q^{(1)} + 2\nu^{(1)}N^{(1)}\cdot\big(\nabla u^{(1)}N^{(1)}\big)\big|_{S_{F,T}^{(1)}} \\ \quad = \bar{b}^{(1)} - 2\nu^{(1)}N^{(1)}\cdot\big[\nabla\big(w^{(1)} + \nabla\varphi^{(1)}\big)N^{(1)}\big] + \Psi^{(1)}\big|_{S_{F,T}^{(1)}} \stackrel{\text{def}}{=} d^{(1)}, \\ \Pi^{(2)}\big[\nu^{(2)}\mathrm{D}\big(u^{(2)}\big)N^{(2)} - \nu^{(1)}\mathrm{D}\big(u^{(1)}\big)N^{(2)}\big]\big|_{S_{F,T}^{(2)}} \\ \quad = \vec{b}^{(2)} - \Pi^{(2)}\Bigg[\sum_{j=1}^{2}(-1)^j\nu^{(j)}\mathrm{D}\big(\nabla\varphi^{(j)} + w^{(j)}\big)N^{(2)}\Bigg]\Bigg|_{S_{F,T}^{(2)}} \stackrel{\text{def}}{=} \vec{d}^{(2)}, \\ q^{(1)} - q^{(2)} + 2\nu^{(2)}N^{(2)}\cdot\big(\nabla u^{(2)}N^{(2)}\big) - 2\nu^{(1)}N^{(2)}\cdot\big(\nabla u^{(1)}N^{(2)}\big)\big|_{S_{F,T}^{(2)}} \\ \quad = \bar{b}^{(2)} - \sum_{j=1}^{2}(-1)^j\big\{2\nu^{(j)}N^{(2)}\cdot\big[\nabla\big(\nabla\varphi^{(j)} + w^{(j)}\big)N^{(2)}\big] - \Psi^{(j)}\big\}\big|_{S_{F,T}^{(2)}} \stackrel{\text{def}}{=} d^{(2)}, \\ u^{(1)} - u^{(2)}\big|_{S_{F,T}^{(2)}} = 0, \quad u^{(2)}\big|_{S_{B,T}} = 0. \end{cases} \tag{3.64}$$

Thanks to (3.60)–(3.62), we obtain by Lemma 2.1 that

$$\sum_{j=1}^{2}\left[\left\|\Phi^{(j)}\right\|_{H_\gamma^{l,l/2}(Q_T^{(j)})}+\left\|\vec{d}^{(j)}\right\|_{H_\gamma^{l+1/2,l/2+1/4}(S_{F,T}^{(j)})}+\left\|d^{(j)}\right\|_{H_\gamma^{l+1/2,1/2,l/2}(S_{F,T}^{(j)})}\right]\leqslant C\Xi_1. \tag{3.65}$$

We define

$$H_{\gamma,N^{(j)}}^{l+1/2,l/2+1/4}\big(S_{F,T}^{(j)}\big)\stackrel{\text{def}}{=}\big\{f\in H_\gamma^{l+1/2,l/2+1/4}\big(S_{F,T}^{(j)}\big)\ \big|\ f\cdot N^{(j)}=0\big\}.$$

In what follows, we shall prove the solvability of (3.63)–(3.64) by constructing a continuous linear operator:

$$\begin{aligned}\mathscr{R}:\mathbb{W}&\stackrel{\text{def}}{=}H_\gamma^{l,l/2}\big(Q_T^{(1)}\big)\times H_{\gamma,N^{(1)}}^{l+1/2,l/2+1/4}\big(S_{F,T}^{(1)}\big)\times H_\gamma^{l+1/2,1/2,l/2}\big(S_{F,T}^{(1)}\big)\\&\quad\times H_\gamma^{l,l/2}\big(Q_T^{(2)}\big)\times H_{\gamma,N^{(2)}}^{l+1/2,l/2+1/4}\big(S_{F,T}^{(2)}\big)\times H_\gamma^{l+1/2,1/2,l/2}\big(S_{F,T}^{(2)}\big)\\&\to H_\gamma^{l+2,l/2+1}\big(Q_T^{(1)}\big)\times H_\gamma^{l+1,1,l/2}\big(Q_T^{(1)}\big)\times H_\gamma^{l+2,l/2+1}\big(Q_T^{(2)}\big)\times H_\gamma^{l+1,1,l/2}\big(Q_T^{(2)}\big)\end{aligned}$$

such that $(u^{(1)},q^{(1)},u^{(2)},q^{(2)})=\mathscr{R}G$ satisfies (3.63)–(3.64) with $(\Phi^{(j)},\vec{d}^{(j)},d^{(j)})$ there being replaced by $(\Phi^{(j)}+\mathscr{N}_{j1}G,\vec{d}^{(j)}+\mathscr{N}_{j2}G,d^{(j)}+\mathscr{N}_{j3}G)$ where $G=(\Phi^{(1)},\vec{d}^{(1)},d^{(1)},\Phi^{(2)},\vec{d}^{(2)},d^{(2)})$ and $\mathscr{N}=(\mathscr{N}_{11},\ \mathscr{N}_{12},\mathscr{N}_{13},\mathscr{N}_{21},\mathscr{N}_{22},\mathscr{N}_{23})$ is a continuous linear operator on $\mathbb{W}$. Then we shall show that $\mathscr{N}$ becomes a contraction if $\gamma\gg 1$, which implies the solvability of (3.63)–(3.64) and the solution $(u^{(1)},q^{(1)},u^{(2)},q^{(2)})=\mathscr{R}(I+\mathscr{N})^{-1}G$.

We cover $\bar{\Omega}$ with $\{\omega_k\}$ as in Section 3.2. Let $\{\zeta_k\}$ be a smooth partition of unit subordinate to $\{\omega_k\}$, i.e.

$$\sum_k\zeta_k=1,\quad 0\leqslant\zeta_k\leqslant 1,\qquad\left|D^\alpha\zeta_k\right|\leqslant C_\alpha|\lambda|^{-|\alpha|},$$

and let $\{\eta_k\}$ be smooth functions satisfying

$$\eta_k(x)=\begin{cases}1, & x\in\omega_k,\\ 0, & x\in\bar{\Omega}\setminus\frac{3}{2}\omega_k,\end{cases}$$

where $\frac{3}{2}\omega_k$ is the cubic with the same center as ω_k and with the 3/2 times length of ω_k.

For the reduced problem (3.63)–(3.64), we construct

$$\begin{aligned}\mathscr{R}G&=\big(u^{(11)},q^{(11)},u^{(21)},q^{(21)}\big)+\big(u^{(12)},0,u^{(22)},0\big)\\&=\sum_k\big(\zeta_k u^{(1)(k)},\zeta_k q^{(1)(k)},\zeta_k u^{(2)(k)},\zeta_k q^{(2)(k)}\big)+\big(u^{(21)},0,u^{(22)},0\big).\end{aligned}$$

For $k=k'$, $(u^{(j)(k')},q^{(j)(k')})$ satisfies the equation

$$\begin{cases}\rho_0^{(j)}\big(\xi^{(k')}\big)u_t^{(j)(k')}-\nu^{(j)}\nabla_x^2u^{(j)(k')}+\nabla_x q^{(j)(k')}=\eta_{k'}\Phi^{(j)}, & x\in\dfrac{3}{2}\omega_{k'}\cap\Omega^{(j)},\\ \nabla_x\cdot u^{(j)(k')}=0,\quad u^{(j)(k')}\big|_{t=0}=0,\end{cases}$$

whose solvability was presented in Chapter 4 of [15].

For $k=k''$, $(u^{(2)(k'')},q^{(2)(k'')})=0$ and $(u^{(1)(k'')}(x,t),q^{(1)(k'')}(x,t))=({}^tL_{k''}w^{(k'')}(Z_{k''}\circ Y_{k''}(x),t), s^{(k'')}(Z_{k''}\circ Y_{k''}(x),t))$ with $(w^{(k'')},s^{(k'')})$ satisfying the problem

$$\begin{cases}\rho_0^{(1)}(\xi^{(k'')})w_t^{(k'')}-\nu^{(1)}\nabla_{k''}^2w^{(k'')}+\nabla_{k''}s^{(k'')}\\ \quad =J_{k''}L_{k''}(\eta_{k''}\Phi^{(1)})(Y_{k''}^{-1}\circ Z_{k''}^{-1}(z))+\bar\Phi^{(1)(k'')}(z)\quad \text{in } \mathbb{D}_T,\\ \nabla_{k''}\cdot w^{(k'')}=0\quad \text{in } \mathbb{D}_T,\quad w^{(k'')}\big|_{t=0}=0\quad \text{on } \mathbb{R}^3_+,\\ \nu^{(1)}\tilde\Pi_{k''}\big[\mathrm{D}_{k''}(w^{(k'')})e_3\big]\big|_{z_3=0}=L_{k''}(\eta_{k''}\vec d^{(1)})(Y_{k''}^{-1}\circ Z_{k''}^{-1}(z))\quad \text{on } \mathbb{R}^2_T,\\ -s^{(k'')}+2\nu^{(1)}e_3\cdot\dfrac{\partial w^{(k'')}}{\partial e_3}\Big|_{z_3=0}=(\eta_{k''}d^{(1)})(Y_{k''}^{-1}\circ Z_{k''}^{-1}(z))\quad \text{on } \mathbb{R}^2_T,\end{cases}$$

where $e_3={}^t(0,0,-1)$ and

$$\bar\Phi^{(1)(k'')}(z)=\frac{1}{4\pi}\int\limits_{\mathbb{R}^3_+}\nabla_z\frac{1}{|z-\tilde z|}(\nabla_{\tilde z}\cdot J_{k''})\cdot\big(L_{k''}(\eta_{k''}\Phi^{(1)})(Y_{k''}^{-1}\circ Z_{k''}^{-1}(\tilde z))\big)\,d\tilde z.$$

For $k=k'''$, $(u^{(j)(k''')}(x,t),q^{(j)(k''')}(x,t))=({}^tL_{k'''}w^{(j)(k''')}(Z_{k'''}\circ Y_{k'''}(x),t), s^{(j)(k''')}(Z_{k'''}\circ Y_{k'''}(x),t))$ with $(w^{(1)(k''')},s^{(1)(k''')},w^{(2)(k''')},s^{(2)(k''')})$ satisfying the problem

$$\begin{cases}\rho_0^{(j)}(\xi^{(k''')})w_t^{(j)(k''')}-\nu^{(j)}\nabla_{k'''}^2w^{(j)(k''')}+\nabla_{k'''}s^{(j)(k''')}\\ \quad =J_{k'''}L_{k'''}(\eta_{k'''}\Phi^{(j)})(Y_{k'''}^{-1}\circ Z_{k'''}^{-1}(z))+\bar\Phi^{(j)(k''')}(z)\quad \text{in } \mathbb{D}_T^{(j)},\\ \nabla_{k'''}\cdot w^{(j)(k''')}=0\quad \text{in } \mathbb{D}_T^{(j)},\\ w^{(1)(k''')}\big|_{t=0}=0\quad \text{on } \mathbb{R}^3_+,\quad w^{(2)(k''')}\big|_{t=0}=0\quad \text{on } \mathbb{R}^3_-,\\ w^{(1)(k''')}-w^{(2)(k''')}\big|_{z_3=0}=0\quad \text{on } \mathbb{R}^2_T,\\ \tilde\Pi_{k'''}\big[\nu^{(2)}\mathrm{D}_{k'''}(w^{(2)(k''')})e_3-\nu^{(1)}\mathrm{D}_{k'''}(w^{(1)(k''')})e_3\big]\big|_{z_3=0}\\ \quad =L_{k'''}(\eta_{k'''}\vec d^{(2)})(Y_{k'''}^{-1}\circ Z_{k'''}^{-1}(z))\quad \text{on } \mathbb{R}^2_T,\\ s^{(1)(k''')}-s^{(2)(k''')}+2\nu^{(2)}e_3\cdot\dfrac{\partial w^{(2)(k''')}}{\partial e_3}-2\nu^{(1)}e_3\cdot\dfrac{\partial w^{(1)(k''')}}{\partial e_3}\Big|_{z_3=0}\\ \quad =(\eta_{k'''}d^{(2)})(Y_{k'''}^{-1}\circ Z_{k'''}^{-1}(z))\quad \text{on } \mathbb{R}^2_T,\end{cases}$$

where $e_3={}^t(0,0,1)$ and

$$\bar\Phi^{(1)(k''')}(z)=\frac{1}{4\pi}\int\limits_{\mathbb{R}^3_+}\nabla_z\frac{1}{|z-\tilde z|}(\nabla_{\tilde z}\cdot J_{k'''})\cdot\big(L_{k'''}(\eta_{k'''}\Phi^{(1)})(Y_{k'''}^{-1}\circ Z_{k'''}^{-1}(\tilde z))\big)\,d\tilde z,$$

$$\bar\Phi^{(2)(k''')}(z)=\frac{1}{4\pi}\int\limits_{\mathbb{R}^3_-}\nabla_z\frac{1}{|z-\tilde z|}(\nabla_{\tilde z}\cdot J_{k'''})\cdot\big(L_{k'''}(\eta_{k'''}\Phi^{(2)})(Y_{k'''}^{-1}\circ Z_{k'''}^{-1}(\tilde z))\big)\,d\tilde z.$$

For $k=k^{(4)}$, $(u^{(1)(k^{(4)})},q^{(1)(k^{(4)})})=0$ and $(u^{(2)(k^{(4)})},q^{(2)(k^{(4)})})$ are defined in the same way as $(u^{(1)(k'')},q^{(1)(k'')})$ with the boundary conditions replaced by $w^{(k^{(4)})}=0$ on $z_3=0$. And $u^{(j2)}$ are constructed in the way of the beginning of this subsection, in order to reduce $\nabla \cdot (u^{(j1)} + u^{(j2)}) = 0$, $j = 1, 2$. Note that

$$\nabla_z \cdot \big(J_k L_k\big(\eta_k \Phi^{(j)}\big)\big(Y_k^{-1} \circ Z_k^{-1}(z)\big) + \bar{\Phi}^{(j)(k)}\big) = \nabla_x \cdot \Phi^{(j)} = 0, \quad \text{in } \omega_k,$$

for $j = 1, k = k'', k'''$ and for $j = 2, k = k'''$. This ensure that we can get the estimate of $\|s^{(j)}\|_{H_\gamma^{0,l/2+1/4}}$ as in the proof of Theorem 3.3.

Therefore, the operator $\mathscr{N} G$ is defined similarly as $\mathscr{M} D$. The only difference is that we have the following additional terms in $\mathscr{N}_{11} G$ and $\mathscr{N}_{21} G$,

$$\sum_{k=k'',k'''} \big({}^t L_k J_k L_k - I\big)\zeta_k \Phi^{(1)}, \qquad \sum_{k=k''',k^{(4)}} \big({}^t L_k J_k L_k - I\big)\zeta_k \Phi^{(2)}.$$

Furthermore, in the same way as the proof of Theorem 3.3, we can show that $\mathscr{N}$ is a contraction if we take $\lambda \ll 1$ and then $\gamma \gg 1$. This completes the proof of Theorem 3.2.

4. Local existence

This section is devoted to the proof of Theorem 1.1 by using the methods in [19,21]. From Theorem 3.1, we can easily conclude that there exists a unique solution $\bar{u}^{(j)} \in W_2^{l+2,l/2+1}(Q_T^{(j)})$, $\nabla \bar{q}^{(j)}, \bar{q}^{(j)} \in W_2^{l,l/2}(Q_T^{(j)})$ for any $T > 0$ to the problem (3.1)–(3.2) with $(f^{(j)}, g^{(j)}, v_0^{(j)}, \vec{b}^{(j)}, B^{(j)}) \equiv 0$, $\bar{b}^{(j)} = -g_0(\xi_3 - h^{(j)})$ such that

$$\begin{aligned} &\sum_{j=1}^{2}\big(\|\bar{u}^{(j)}\|_{W_2^{l+2,l/2+1}(Q_T^{(j)})} + \|\nabla\bar{q}^{(j)}\|_{W_2^{l,l/2}(Q_T^{(j)})} + \|\bar{q}^{(j)}\|_{W_2^{l,l/2}(Q_T^{(j)})}\big) \\ &\qquad + \|\bar{q}^{(1)}\|_{W_2^{l+1/2,l/2+1/4}(S_{F,T}^{(1)})} + \|\bar{q}^{(1)} - \bar{q}^{(2)}\|_{W_2^{l+1/2,l/2+1/4}(S_{F,T}^{(2)})} \\ &\quad \leqslant C(T)\sum_{j=1}^{2}\|F_0^{(j)} - h^{(j)}\|_{W_2^{l+1/2}(\mathbb{R}^2)}. \end{aligned} \tag{4.1}$$

We seek a solution to the problem (1.8)–(1.9) in the form

$$\big(u^{(j)}, q^{(j)}\big) = \big(\bar{u}^{(j)} + \tilde{u}^{(j)}, \bar{q}^{(j)} + \tilde{q}^{(j)}\big), \quad j = 1, 2.$$

Then the problem (1.8)–(1.9) is reduced to the following problem for $(\tilde{u}^{(j)}, \tilde{q}^{(j)})$

$$\begin{cases} \rho_0^{(j)}\tilde{u}_t^{(j)} - \nu^{(j)}\nabla_{u^{(j)}}^2\tilde{u}^{(j)} + \nabla_{u^{(j)}}\tilde{q}^{(j)} = \rho_0^{(j)} f^{(j)}\big(X_{u^{(j)}}(\xi, t), t\big) \\ \quad + \nu^{(j)}\big(\nabla_{u^{(j)}}^2 - \nabla^2\big)\bar{u}^{(j)} - (\nabla_{u^{(j)}} - \nabla)\bar{q}^{(j)}, \quad \xi \in \Omega^{(j)},\ t > 0, \\ \nabla_{u^{(j)}} \cdot \tilde{u}^{(j)} = -\nabla_{u^{(j)}} \cdot \bar{u}^{(j)}, \quad \xi \in \Omega^{(j)},\ t > 0, \\ \tilde{u}^{(j)}\big|_{t=0} = v_0^{(j)}(\xi), \quad \xi \in \Omega^{(j)},\ j = 1, 2, \end{cases} \tag{4.2}$$

together with the boundary conditions

$$\begin{cases} \Pi^{(1)}_{u^{(1)}}\big[\nu^{(1)}\mathrm{D}_{u^{(1)}}\big(\tilde{u}^{(1)}\big)n^{(1)}_{u^{(1)}}\big] = -\Pi^{(1)}_{u^{(1)}}\big[\nu^{(1)}\mathrm{D}_{u^{(1)}}\big(\bar{u}^{(1)}\big)n^{(1)}_{u^{(1)}}\big], \quad \xi \in S^{(1)}_F,\ t>0, \\ -\tilde{q}^{(1)} + 2\nu^{(1)}n^{(1)}_{u^{(1)}}\cdot\big(\nabla_{u^{(1)}}\tilde{u}^{(1)}n^{(1)}_{u^{(1)}}\big) \\ \quad = \sigma^{(1)}n^{(1)}_{u^{(1)}}\cdot\Delta^{(1)}_{u^{(1)}}(t)X_{u^{(1)}} - 2\nu^{(1)}n^{(1)}_{u^{(1)}}\cdot\big(\nabla_{u^{(1)}}\bar{u}^{(1)}n^{(1)}_{u^{(1)}}\big) \\ \qquad + 2\nu^{(1)}N^{(1)}\cdot\big(\nabla\bar{u}^{(1)}N^{(1)}\big) - g_0\displaystyle\int_0^t u^{(1)}_3\,d\tau - \sigma^{(1)}N^{(1)}\cdot\Delta^{(1)}\int_0^t \bar{u}^{(1)}\,d\tau, \\ \tilde{u}^{(1)} - \tilde{u}^{(2)} = 0, \quad \xi \in S^{(2)}_F,\ t>0, \\ \Pi^{(2)}_{u^{(2)}}\big[\nu^{(2)}\mathrm{D}_{u^{(2)}}\big(\tilde{u}^{(2)}\big)n^{(2)}_{u^{(2)}}\big] - \Pi^{(2)}_{u^{(1)}}\big[\nu^{(1)}\mathrm{D}_{u^{(1)}}\big(\tilde{u}^{(1)}\big)n^{(2)}_{u^{(1)}}\big] \\ \quad = \Pi^{(2)}_{u^{(1)}}\big[\nu^{(1)}\mathrm{D}_{u^{(1)}}\big(\bar{u}^{(1)}\big)n^{(2)}_{u^{(1)}}\big] - \Pi^{(2)}_{u^{(2)}}\big[\nu^{(2)}\mathrm{D}_{u^{(2)}}\big(\bar{u}^{(2)}\big)n^{(2)}_{u^{(2)}}\big], \quad \xi \in S^{(2)}_F,\ t>0, \\ \tilde{q}^{(1)} - \tilde{q}^{(2)} + 2\nu^{(2)}n^{(2)}_{u^{(2)}}\cdot\big(\nabla_{u^{(2)}}\tilde{u}^{(2)}n^{(2)}_{u^{(2)}}\big) - 2\nu^{(1)}n^{(2)}_{u^{(1)}}\cdot\big(\nabla_{u^{(1)}}\tilde{u}^{(1)}n^{(2)}_{u^{(1)}}\big) \\ \quad = \dfrac{\sigma^{(2)}}{2}\displaystyle\sum_{j=1}^2 n^{(2)}_{u^{(j)}}\cdot\Delta^{(2)}_{u^{(j)}}(t)X_{u^{(j)}} - \dfrac{\sigma^{(2)}}{2}\sum_{j=1}^2 N^{(2)}\cdot\Delta^{(2)}\int_0^t \bar{u}^{(j)}\,d\tau \\ \qquad - 2\displaystyle\sum_{j=1}^2(-1)^j\nu^{(j)}\big[n^{(2)}_{u^{(j)}}\cdot\big(\nabla_{u^{(j)}}\bar{u}^{(j)}n^{(2)}_{u^{(j)}}\big) - N^{(2)}\cdot\big(\nabla\bar{u}^{(j)}N^{(2)}\big)\big] - g_0\int_0^t u^{(2)}_3\,d\tau, \\ \tilde{u}^{(2)} = 0, \quad \xi \in S_B,\ t>0, \end{cases} \tag{4.3}$$

where $u^{(j)} = \bar{u}^{(j)} + \tilde{u}^{(j)}$. Notice that

$$\begin{aligned} n^{(1)}_{u^{(1)}}\cdot\Delta^{(1)}_{u^{(1)}}X_{u^{(1)}} = {} & H^{(1)}_0(\xi) + N^{(1)}\cdot\Delta^{(1)}_0\int_0^t\big(\bar{u}^{(1)}+\tilde{u}^{(1)}\big)\,d\tau \\ & + \int_0^t\bigg[\big(n^{(1)}_{u^{(1)}}\cdot\Delta^{(1)}_{u^{(1)}}(\tau) - N^{(1)}\cdot\Delta^{(1)}_0\big)u^{(1)} \\ & + \bigg(\frac{\partial}{\partial\tau}n^{(1)}_{u^{(1)}}\cdot\Delta^{(1)}_{u^{(1)}}(\tau) + n^{(1)}_{u^{(1)}}\cdot\dot{\Delta}^{(1)}_{u^{(1)}}(\tau)\bigg)\bigg(\xi + \int_0^\tau u^{(1)}ds\bigg)\bigg]\,d\tau, \end{aligned}$$

where $\Delta^{(1)}_0 = \Delta^{(1)} = \Delta^{(1)}(0)$, $H^{(1)}_0 = N^{(1)}\cdot\Delta^{(1)}_0\xi$ is twice the mean curvature of $S^{(1)}_F$ and $\dot{\Delta}^{(1)}_{u^{(1)}}(t)$ is an operator obtained from $\Delta^{(1)}_{u^{(1)}}$ by differentiating its coefficients with respect to t. $n^{(2)}_{u^{(j)}}\cdot\Delta^{(2)}_{u^{(j)}}X_{u^{(j)}}$ has the same representation with the corresponding change of the index. Thus, we can rewrite the boundary conditions $(4.3)_2$ and $(4.3)_5$ in the form

$$\begin{aligned} & -\tilde{q}^{(1)} + 2\nu^{(1)}N^{(1)}\cdot\big(\nabla\tilde{u}^{(1)}N^{(1)}\big) - \sigma^{(1)}\Delta^{(1)}\int_0^t\tilde{u}^{(1)}\,d\tau\cdot N^{(1)} \\ & \quad = \sigma^{(1)}H^{(1)}_0(\xi) + 2\nu^{(1)}\big[N^{(1)}\cdot\big(\nabla u^{(1)}N^{(1)}\big) - n^{(1)}_{u^{(1)}}\cdot\big(\nabla_{u^{(1)}}u^{(1)}n^{(1)}_{u^{(1)}}\big)\big] \\ & \qquad + \sigma^{(1)}\int_0^t\bigg[\big(n^{(1)}_{u^{(1)}}\cdot\Delta^{(1)}_{u^{(1)}}(\tau) - N^{(1)}\cdot\Delta^{(1)}_0\big)u^{(1)} + \bigg(\frac{\partial}{\partial\tau}n^{(1)}_{u^{(1)}}\cdot\Delta^{(1)}_{u^{(1)}}(\tau) \end{aligned}$$

$$+ n^{(1)}_{u^{(1)}} \cdot \dot{\Delta}^{(1)}_{u^{(1)}}(\tau)\Bigg)\Bigg(\xi + \int_0^\tau u^{(1)}\,ds\Bigg)\Bigg]d\tau - g_0\int_0^t u_3^{(1)}\,d\tau$$

$$\stackrel{\text{def}}{=} b^{(1)}\big(u^{(1)}\big) + \sigma^{(1)}\int_0^t B^{(1)}\big(u^{(1)}\big)\,d\tau - g_0\int_0^t u_3^{(1)}\,d\tau,$$

and

$$\tilde q^{(1)} - \tilde q^{(2)} + 2\nu^{(2)}N^{(2)}\cdot\big(\nabla\tilde u^{(2)}N^{(2)}\big) - 2\nu^{(1)}N^{(2)}\cdot\big(\nabla\tilde u^{(1)}N^{(2)}\big)$$

$$- \frac{\sigma^{(2)}}{2}N^{(2)}\cdot\Delta^{(2)}\int_0^t\big(\tilde u^{(1)} + \tilde u^{(2)}\big)\,ds$$

$$= \sigma^{(2)}H_0^{(2)}(\xi) + 2\sum_{j=1}^{2}(-1)^j\nu^{(j)}\big[N^{(2)}\cdot\big(\nabla u^{(j)}N^{(2)}\big) - n^{(2)}_{u^{(j)}}\cdot\big(\nabla_{u^{(j)}}u^{(j)}n^{(2)}_{u^{(j)}}\big)\big]$$

$$+ \frac{\sigma^{(2)}}{2}\int_0^t\sum_{j=1}^{2}\Bigg[\big(n^{(2)}_{u^{(j)}}\cdot\Delta^{(2)}_{u^{(j)}}(\tau) - N^{(2)}\cdot\Delta_0^{(2)}\big)u^{(j)} + \Bigg(\frac{\partial}{\partial\tau}n^{(2)}_{u^{(j)}}\cdot\Delta^{(2)}_{u^{(j)}}(\tau)$$

$$+ n^{(2)}_{u^{(j)}}\cdot\dot{\Delta}^{(2)}_{u^{(j)}}(\tau)\Bigg)\Bigg(\xi + \int_0^\tau u^{(j)}\,ds\Bigg)\Bigg]d\tau - g_0\int_0^t u_3^{(2)}\,d\tau$$

$$\stackrel{\text{def}}{=} b^{(2)}\big(u^{(1)}, u^{(2)}\big) + \frac{\sigma^{(2)}}{2}\int_0^t B^{(2)}\big(u^{(1)}, u^{(2)}\big)\,d\tau - g_0\int_0^t u_3^{(2)}\,d\tau.$$

4.1. Some technical lemmas

Lemma 4.1. *Assume that $u, u' \in W_2^{l+2,l/2+1}(Q_T)$ satisfying*

$$T^{1/2-l/2}\big\|\big(u, u'\big)\big\|_{W_2^{l+2,l/2+1}(Q_T)} \leqslant \delta, \tag{4.4}$$

with a small enough positive constant δ. Then there hold for $t \leqslant T$,

(i) $\|\mathcal{A}_u - \mathcal{A}_{u'}\|_{L^\infty(0,t;W_2^{l+1}(\Omega))} \leqslant C(t,\delta)t^{1/2}\big\|u - u'\big\|_{W_2^{l+2,l/2+1}(Q_t)}$;

(ii) $\|\mathcal{A}_u - \mathcal{A}_{u'}\|_{L^\infty(\Omega;W_2^{(l+1)/2}(0,t))} \leqslant C(t,\delta)t^{1/2-l/2}\big\|u - u'\big\|_{W_2^{l+2,l/2+1}(Q_t)}$;

(iii) $\|\mathcal{A}_u - \mathcal{A}_{u'}\|_{W_2^{l+1,l/2+1/2}(Q_t)} \leqslant C(t,\delta)t^{1-l/2}\big\|u - u'\big\|_{W_2^{l+2,l/2+1}(Q_t)}$;

(iv) $\left\|\frac{\partial}{\partial t}(\mathcal{A}_u - \mathcal{A}_{u'})\right\|_{W_2^{l+1,l/2+1/2}(Q_t)} \leqslant C(t,\delta)\big\|u - u'\big\|_{W_2^{l+2,l/2+1}(Q_t)}$.

Throughout this section, we denote by $C(t,\delta)$ a positive constant depending increasingly on t and δ.

Proof. Since $\mathcal{A}_u = \mathrm{Det}(\frac{\partial X_u}{\partial \xi})^{-1} {}^t(\frac{\partial X_u}{\partial \xi})^{-1} \stackrel{\text{def}}{=} \{G_{jk}^{(u)}\}_{j,k=1,2,3}$ the adjugate matrix of ${}^t(\frac{\partial X_u}{\partial \xi})$, it suffices to prove (i)–(iv) for $G_{jk}^{(u)}$. Let us first prove (iv). Due to the definition of $G_{jk}^{(u)}$, we have

$$\begin{aligned}
&\left\| \frac{\partial}{\partial t}\big(G_{jk}^{(u)} - G_{jk}^{(u')}\big)\right\|_{W_2^{l+1}(\Omega)} \\
&\quad\leqslant C\big\|\nabla u - \nabla u'\big\|_{W_2^{l+1}(\Omega)}\left(1 + \int_0^t \|\nabla u\|_{W_2^{l+1}(\Omega)}\,d\tau + \int_0^t \big\|\nabla u'\big\|_{W_2^{l+1}(\Omega)}\,d\tau\right) \\
&\qquad + C\big(\|\nabla u\|_{W_2^{l+1}(\Omega)} + \big\|\nabla u'\big\|_{W_2^{l+1}(\Omega)}\big)\int_0^t \big\|\nabla u - \nabla u'\big\|_{W_2^{l+1}(\Omega)}\,d\tau \\
&\quad\leqslant C\big\{\big\|\nabla u - \nabla u'\big\|_{W_2^{l+1}(\Omega)}\big(1 + t^{1/2}\|u\|_{W_2^{l+2,0}(Q_T)} + t^{1/2}\big\|u'\big\|_{W_2^{l+2,0}(Q_t)}\big) \\
&\qquad + \big(\|u\|_{W_2^{l+2}(\Omega)} + \big\|u'\big\|_{W_2^{l+2}(\Omega)}\big)t^{1/2}\big\|u - u'\big\|_{W_2^{l+2,0}(Q_t)}\big\},
\end{aligned}$$

from which and (4.4), we get

$$\left\| \frac{\partial}{\partial t}\big(G_{jk}^{(u)} - G_{jk}^{(u')}\big)\right\|_{W_2^{l+1,0}(Q_t)} \leqslant C(t,\delta)\big\|u - u'\big\|_{W_2^{l+2,0}(Q_t)}. \tag{4.5}$$

Similarly, we have

$$\begin{aligned}
&\left\| \frac{\partial}{\partial t}\big(G_{jk}^{(u)} - G_{jk}^{(u')}\big)\right\|_{W_2^{l/2+1/2}(0,t)} \\
&\quad\leqslant C\big\|\nabla u - \nabla u'\big\|_{W_2^{l/2+1/2}(0,t)}\left(1 + \left\|\int_0^t \nabla u\,d\tau\right\|_{W_2^{l/2+1/2}(0,t)} + \left\|\int_0^t \nabla u'\,d\tau\right\|_{W_2^{l/2+1/2}(0,t)}\right) \\
&\qquad + C\big(\|\nabla u\|_{W_2^{l/2+1/2}(0,t)} + \big\|\nabla u'\big\|_{W_2^{l/2+1/2}(0,t)}\big)\left\|\int_0^t \big(\nabla u - \nabla u'\big)\,d\tau\right\|_{W_2^{l/2+1/2}(0,t)}
\end{aligned}$$

and note that

$$\begin{aligned}
\left\|\int_0^t \nabla u\,d\tau\right\|_{L^2(0,t)} &\leqslant t\|\nabla u\|_{L^2(0,t)} \leqslant Ct\|u\|_{W_2^{l+2,0}(Q_t)}, \\
\left\|\int_0^t \nabla u\,d\tau\right\|^2_{\dot{W}_2^{l/2+1/2}(0,t)} &\leqslant 2\int_0^t\int_0^s \frac{|\int_{s-\tau}^s \nabla u\,dr|^2}{\tau^{l+2}}\,d\tau\,ds \\
&\leqslant C\int_0^t\int_0^s \frac{\int_{s-\tau}^s |\nabla u|^2\,dr}{\tau^{l+1}}\,d\tau\,ds \\
&\leqslant Ct^{1-l}\|\nabla u\|^2_{L^2(0,t)} \quad \big(\text{or } Ct^{2-l}\|\nabla u\|^2_{L^\infty(0,t)}\big),
\end{aligned}$$

whence

$$\left\| \frac{\partial}{\partial t}\left(G_{jk}^{(u)} - G_{jk}^{(u')}\right)\right\|_{W_2^{0,l/2+1/2}(Q_t)} \leqslant C(t,\delta)\left\|u - u'\right\|_{W_2^{l+2,l/2+1}(Q_t)},$$

which together with (4.5) gives

$$\left\| \frac{\partial}{\partial t}\left(G_{jk}^{(u)} - G_{jk}^{(u')}\right)\right\|_{W_2^{l+1,l/2+1/2}(Q_t)} \leqslant C(t,\delta)\left\|u - u'\right\|_{W_2^{l+2,l/2+1}(Q_t)}. \tag{4.6}$$

This proves (iv). Next, we use (4.6) to prove (i)–(iii). Since

$$G_{jk}^{(u)} - G_{jk}^{(u')} = \int_0^t \frac{\partial}{\partial \tau}\left(G_{jk}^{(u)} - G_{jk}^{(u')}\right) d\tau,$$

we can deduce from (4.6) that

$$\begin{aligned}\left\|G_{jk}^{(u)} - G_{jk}^{(u')}\right\|_{L^\infty(0,t;W_2^{l+1}(\Omega))} &\leqslant C(t,\delta)t^{1/2}\left\|u - u'\right\|_{W_2^{l+2,l/2+1}(Q_t)},\\ \left\|G_{jk}^{(u)} - G_{jk}^{(u')}\right\|_{W_2^{l+1,0}(Q_t)} &\leqslant C(t,\delta)t\left\|u - u'\right\|_{W_2^{l+2,l/2+1}(Q_t)},\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}&\left\|G_{jk}^{(u)} - G_{jk}^{(u')}\right\|^2_{L^2(0,t)}\\ &\quad\leqslant Ct^2\left\| \frac{\partial}{\partial t}\left(G_{jk}^{(u)} - G_{jk}^{(u')}\right)\right\|^2_{W_2^{l+1,0}(Q_t)} \quad \left(\text{or } Ct^3\left\| \frac{\partial}{\partial t}\left(G_{jk}^{(u)} - G_{jk}^{(u')}\right)\right\|^2_{W_2^{l/2+1/2}(0,t)}\right),\\ &\left\|G_{jk}^{(u)} - G_{jk}^{(u')}\right\|^2_{\dot W_2^{l/2+1/2}(0,t)}\\ &\quad\leqslant Ct^{1-l}\left\| \frac{\partial}{\partial t}\left(G_{jk}^{(u)} - G_{jk}^{(u')}\right)\right\|^2_{W_2^{l+1,0}(Q_t)} \quad \left(\text{or } Ct^{2-l}\left\| \frac{\partial}{\partial t}\left(G_{jk}^{(u)} - G_{jk}^{(u')}\right)\right\|^2_{W_2^{l/2+1/2}(0,t)}\right),\end{aligned}$$

that is

$$\begin{aligned}\left\|G_{jk}^{(u)} - G_{jk}^{(u')}\right\|_{L^\infty(\Omega;W_2^{l/2+1/2}(0,t))} &\leqslant C(t,\delta)t^{1/2-l/2}\left\|u - u'\right\|_{W_2^{l+2,l/2+1}(Q_t)},\\ \left\|G_{jk}^{(u)} - G_{jk}^{(u')}\right\|_{W_2^{0,l/2+1/2}(Q_t)} &\leqslant C(t,\delta)t^{1-l/2}\left\|u - u'\right\|_{W_2^{l+2,l/2+1}(Q_t)}.\end{aligned} \tag{4.8}$$

From (4.7)–(4.8), we can conclude the proof of (i)–(iii). □

Remark 4.1. From the definition of $G_{jk}^{(u)}$, it is easy to get that

$$\left\| \frac{\partial}{\partial x_i} G_{jk}^{(u)}\right\|_{W_2^{l/2+1/2}(0,t)} \leqslant C(t,\delta)t^{1/2-l/2}\|\nabla\nabla u\|_{L^2(0,t)},$$

which implies that

$$\left\| \frac{\partial}{\partial x_i} \mathcal{A}_u \right\|_{W_2^{l/2+1/2}(0,t)} \leqslant C(t,\delta) t^{1/2-l/2} \|\nabla \nabla u\|_{L^2(0,t)}.$$

Lemma 4.2. *Assume that $u, u' \in W_2^{l+2,l/2+1}(Q_T)$ satisfying (4.4). Then there hold for any $t \leqslant T$,*

(i) $\|n_u - n_{u'}\|_{L^\infty(0,t;W_2^{l+1/2}(S_F))} \leqslant C(t,\delta) t^{1/2} \|u - u'\|_{W_2^{l+2,l/2+1}(Q_t)}$;

(ii) $\|n_u - n_{u'}\|_{L^\infty(S_F;W_2^{l/2+1/2}(0,t))} \leqslant C(t,\delta) t^{1/2-l/2} \|u - u'\|_{W_2^{l+2,l/2+1}(Q_t)}$;

(iii) $\|n_u - n_{u'}\|_{W_2^{l+1/2,l/2+1/4}(S_{F,t})} \leqslant C(t,\delta) t^{1-l/2} \|u - u'\|_{W_2^{l+2,l/2+1}(Q_t)}$;

(iv) $\left\| \frac{\partial}{\partial t}(n_u - n_{u'}) \right\|_{W_2^{l+1/2,l/2+1/4}(S_{F,t})} \leqslant C(t,\delta) \|u - u'\|_{W_2^{l+2,l/2+1}(Q_t)}$.

Proof. Using the same notations as that in the proof of Lemma 4.1, we can easily get

$$\begin{aligned} |G_{jk} - \delta_{jk}| &\leqslant C t^{1/2} \|u\|_{W_2^{l+2,0}(Q_t)} \big(1 + t^{1/2} \|u\|_{W_2^{l+2,0}(Q_t)}\big) \\ &\leqslant C T^{l/2} \delta \big(1 + T^{l/2} \delta\big). \end{aligned}$$

So for the fixed T (we can assume $T < 1$), we can take sufficiently small δ such that $1/4 < |\mathcal{A}_u N| < 5/4$. We write

$$n_u - n_{u'} = \frac{1}{|\mathcal{A}_u N|} (\mathcal{A}_u - \mathcal{A}_{u'}) N + \frac{|\mathcal{A}_{u'} N| - |\mathcal{A}_u N|}{|\mathcal{A}_u N| |\mathcal{A}_{u'} N|} \mathcal{A}_{u'} N.$$

First of all, we have

$$\left\| \frac{1}{|\mathcal{A}_u N|} (\mathcal{A}_u - \mathcal{A}_{u'}) N \right\|_{W_2^{l+1}(\Omega)} \leqslant C(t,\delta) \|\mathcal{A}_u - \mathcal{A}_{u'}\|_{W_2^{l+1}(\Omega)},$$

$$\left\| \frac{1}{|\mathcal{A}_u N|} (\mathcal{A}_u - \mathcal{A}_{u'}) N \right\|_{W_2^{l/2+1/2}(0,t)} \leqslant C(t,\delta) \|\mathcal{A}_u - \mathcal{A}_{u'}\|_{W_2^{l/2+1/2}(0,t)},$$

from which and Lemma 4.1, we obtain

$$\left\| \frac{1}{|\mathcal{A}_u N|} (\mathcal{A}_u - \mathcal{A}_{u'}) N \right\|_{L^\infty(0,t;W_2^{l+1}(\Omega))} \leqslant C(t,\delta) t^{1/2} \|u - u'\|_{W_2^{l+2,l/2+1}(Q_t)},$$

$$\left\| \frac{1}{|\mathcal{A}_u N|} (\mathcal{A}_u - \mathcal{A}_{u'}) N \right\|_{L^\infty(\Omega;W_2^{l/2+1/2}(0,t))} \leqslant C(t,\delta) t^{1/2-l/2} \|u - u'\|_{W_2^{l+2,l/2+1}(Q_t)},$$

$$\left\| \frac{1}{|\mathcal{A}_u N|} (\mathcal{A}_u - \mathcal{A}_{u'}) N \right\|_{W_2^{l+1,l/2+1/2}(Q_t)} \leqslant C(t,\delta) t^{1-l/2} \|u - u'\|_{W_2^{l+2,l/2+1}(Q_t)}.$$

Similar estimates hold for $\frac{|\mathcal{A}_{u'} N| - |\mathcal{A}_u N|}{|\mathcal{A}_u N| |\mathcal{A}_{u'} N|} \mathcal{A}_{u'} N$. Therefore, we obtain (i)–(iii) by the trace theorem.

Since $\mathrm{Det}(\frac{\partial X_u}{\partial \xi})^{-1}\mathcal{A}_u{}^t(\frac{\partial X_u}{\partial \xi}) = I$ and $n_u = \frac{\mathcal{A}_u N}{|\mathcal{A}_u N|}$, we can deduce that

$$\frac{\partial}{\partial t} n_u = \mathrm{Det}\left(\frac{\partial X_u}{\partial \xi}\right)^{-1}\left[-\mathcal{A}_u{}^t(\nabla u)n_u + n_u\big(n_u \cdot \mathcal{A}_u{}^t(\nabla u)n_u\big)\right],$$

from which, we get (iv) by the trace theorem, Lemma 4.1 and (i)–(iii). □

Lemma 4.3. *Assume that $u, u' \in W_2^{l+2,l/2+1}(Q_T)$ satisfying (4.4) and $u'' \in W_2^{l+2,l/2+1}(Q_T)$. Then there hold for any $t \leqslant T$,*

$$\text{(i)}\quad \left\|\big(\Delta_u(t) - \Delta_{u'}(t)\big)u''\right\|_{W_2^{l-1/2,l/2-1/4}(S_{F,t})} + \left\|\big(\dot{\Delta}_u(t) - \dot{\Delta}_{u'}(t)\big)\int_0^t u''\,d\tau\right\|_{W_2^{l-1/2,l/2-1/4}(S_{F,t})}$$
$$\leqslant C(t,\delta)t^{1/2-l/2}\left\|u - u'\right\|_{W_2^{l+2,l/2+1}(Q_t)}\left\|u''\right\|_{W_2^{l+2,l/2+1}(Q_t)};$$

$$\text{(ii)}\quad \left\|\big(\Delta_u(t) - \Delta_{u'}(t)\big)\int_0^t u''\,d\tau\right\|_{L^\infty(0,t;W_2^{l-1/2}(S_F))}$$
$$\leqslant C(t,\delta)t\left\|u - u'\right\|_{W_2^{l+2,l/2+1}(Q_t)}\left\|u''\right\|_{W_2^{l+2,l/2+1}(Q_t)};$$

$$\text{(iii)}\quad \left\|\big(\Delta_u(t) - \Delta_{u'}(t)\big)\int_0^t u''\,d\tau\right\|_{W_2^{l/2+1/2}(0,t)}$$
$$\leqslant C(t,\delta)t^{1-l}\left\|u - u'\right\|_{W_2^{l+2,l/2+1}(Q_t)}\big(\left\|\nabla u''\right\|_{L^2(0,t)} + \left\|\nabla\nabla u''\right\|_{L^2(0,t)}\big).$$

Proof. Let us represent $\Delta_u(t)$ explicitly on $S_F^{(k)} \equiv S_F \cap \Omega_k = \{y \in \mathbb{R}^3 \mid y' \in \mathbb{R}^2,\ |y'| \leqslant r,\ y_3 = \varphi(y')\}$ for some small $r \in (0,1)$. The relation that between Euler coordinates $\{x\}$ and Cartesian coordinates $\{y\}$

$$x = X_u\big(\xi(y),t\big) = \xi^{(k)} + {}^tL_k y\big|_{y_3=\varphi(y')} + \int_0^t \hat{u}(y,\tau)\,d\tau,$$

with $\hat{u}(y,\tau) = {}^tL_k u(\xi(y),t)$ yields that

$$g_{\alpha\beta} \overset{\text{def}}{=} g_{\alpha\beta}^{(u)} \overset{\text{def}}{=} \frac{\partial x}{\partial y_\alpha}\cdot\frac{\partial x}{\partial y_\beta}$$
$$= \delta_{\alpha\beta} + \frac{\partial\varphi}{\partial y_\alpha}\frac{\partial\varphi}{\partial y_\beta} + \frac{\partial\varphi}{\partial y_\alpha}W_{3\beta} + \frac{\partial\varphi}{\partial y_\beta}W_{3\alpha} + W_{\alpha\beta} + W_{\beta\alpha} + \sum_{j=1}^3 W_{j\alpha}W_{j\beta}, \tag{4.9}$$

where $W_{j\alpha} = W_{j\alpha}^{(u)} = \int_0^t[\frac{\partial\hat{u}_j(y,\tau)}{\partial y_\alpha} + \frac{\partial\hat{u}_j}{\partial y_3}\frac{\partial\varphi}{\partial y_\alpha}]\,d\tau$. We denote $g = g^{(u)} = g_{11}^{(u)}g_{22}^{(u)} - {g_{12}^{(u)}}^2$, $(g^{\alpha\beta})_{\alpha,\beta=1,2} = (g_{\alpha\beta})_{\alpha,\beta=1,2}^{-1}$, $g_{\alpha\beta} = g_{\alpha\beta}^{(u)}$ and $\tilde{g}_{\alpha\beta} = gg^{\alpha\beta}$, $g'_{\alpha\beta} = g_{\alpha\beta}^{(u')}$, $g'^{\alpha\beta} = g^{(u')\alpha\beta}$ etc.

Similar to the proof of Lemma 4.1 and 4.2, we can obtain

$$\begin{aligned}
&\left\|\frac{\partial}{\partial y_\gamma}\big(W_{j\alpha}-W'_{j\alpha}\big)\right\|_{W_2^{l/2+1/2}(0,t)} \\
&\quad\leqslant C\big(t^{1/2-l/2}+t\big)\big(\big\|\nabla\nabla\big(u_j-u'_j\big)\big\|_{L^2(0,t)}+|\nabla\nabla\varphi|\big\|\nabla\big(u_j-u'_j\big)\big\|_{L^2(0,t)}\big),
\end{aligned}\tag{4.10}$$

and

$$\begin{aligned}
\|W_{j\alpha}-W'_{j\alpha}\|_{L^\infty(0,t;W_2^{l+1}(\Omega_k))} &\leqslant Ct^{1/2}\|u-u'\|_{W_2^{l+2,0}(Q_t^{(k)})},\\
\|W_{j\alpha}-W'_{j\alpha}\|_{L^\infty(\Omega_k;W_2^{l/2+1/2}(0,t))} &\leqslant C\big(t^{1/2-l/2}+t^{1/2}\big)\|u-u'\|_{W_2^{l+2,0}(Q_t^{(k)})},\\
\|W_{j\alpha}-W'_{j\alpha}\|_{W_2^{l+1,l/2+1/2}(Q_t^{(k)})} &\leqslant C\big(t+t^{1-l/2}\big)\|u-u'\|_{W_2^{l+2,l/2+1}(Q_t^{(k)})},
\end{aligned}$$

where $Q_t^{(k)}=\Omega_k\times(0,t)$. Thus, we have

$$\begin{aligned}
\|g_{\alpha\beta}-g'_{\alpha\beta}\|_{L^\infty(0,t;W_2^{l+1}(\Omega_k))} &\leqslant C(t,\delta)t^{1/2}\|u-u'\|_{W_2^{l+2,0}(Q_t^{(k)})},\\
\|g_{\alpha\beta}-g'_{\alpha\beta}\|_{L^\infty(\Omega_k;W_2^{l/2+1/2}(0,t))} &\leqslant C(t,\delta)t^{1/2-l/2}\|u-u'\|_{W_2^{l+2,0}(Q_t^{(k)})},\\
\|g_{\alpha\beta}-g'_{\alpha\beta}\|_{W_2^{l+1,l/2+1/2}(Q_t^{(k)})} &\leqslant C(t,\delta)t^{1-l/2}\|u-u'\|_{W_2^{l+2,l/2+1}(Q_t^{(k)})}.
\end{aligned}\tag{4.11}$$

Note that

$$\begin{aligned}
g&=g_{11}g_{22}-g_{12}^2\\
&\geqslant\big(1+|\nabla'\varphi|^2\big)\big[1-C_0\big(\delta+\delta^2+\delta^3\big)\big]>0,
\end{aligned}\tag{4.12}$$

by taking small enough $\delta\leqslant\delta_0\ll1$, (4.11) holds also for $\tilde g_{\alpha\beta}-\tilde g'_{\alpha\beta}$ and $g^{\alpha\beta}-g'^{\alpha\beta}$. Due to (1.3), we get

$$\begin{aligned}
\big(\Delta_u(t)-\Delta_{u'}(t)\big)u''=\sum_{\alpha,\beta=1}^{2}\bigg\{&\big(g^{\alpha\beta}-g'^{\alpha\beta}\big)\frac{\partial^2}{\partial y_\alpha\partial y_\beta}\\
&-\bigg[\frac{1}{\sqrt g}\frac{\partial}{\partial y_\alpha}\big(\sqrt g g^{\alpha\beta}\big)-\frac{1}{\sqrt{g'}}\frac{\partial}{\partial y_\alpha}\big(\sqrt{g'}g'^{\alpha\beta}\big)\bigg]\frac{\partial}{\partial y_\beta}\bigg\}{}^tL_k\hat{u''}.
\end{aligned}$$

For the first term of the right-hand side, we get by Hölder's inequality and (4.11) that

$$\begin{aligned}
&\left\|\big(g^{\alpha\beta}-g'^{\alpha\beta}\big)\frac{\partial^2}{\partial y_\alpha\partial y_\beta}\big({}^tL_k\hat{u''}\big)\right\|_{W_2^{l,l/2}(Q_t^{(k)})}\\
&\quad\leqslant C\big(\|g^{\alpha\beta}-g'^{\alpha\beta}\|_{L^\infty(0,t;W_2^{l+1}(\Omega_k))}+\|g^{\alpha\beta}-g'^{\alpha\beta}\|_{L^\infty(\Omega_k;W_2^{l/2+1/2}(0,t))}\big)\\
&\qquad\times\left\|\frac{\partial^2}{\partial y_\alpha\partial y_\beta}\hat{u''}\right\|_{W_2^{l,l/2}(Q_t^{(k)})}\\
&\quad\leqslant C(t,\delta)t^{1/2-l/2}\|u''\|_{W_2^{l+2,l/2+1}(Q_t^{(k)})}\|u-u'\|_{W_2^{l+2,l/2+1}(Q_t^{(k)})}.
\end{aligned}\tag{4.13}$$

For the second term, we write

$$\left[\frac{1}{\sqrt{g}}\frac{\partial}{\partial y_\alpha}\left(\sqrt{g}g^{\alpha\beta}\right)-\frac{1}{\sqrt{g'}}\frac{\partial}{\partial y_\alpha}\left(\sqrt{g'}g'^{\alpha\beta}\right)\right]\frac{\partial}{\partial y_\beta}\left({}^tL_k\hat{u''}\right)$$
$$=\frac{\partial}{\partial y_\alpha}\left(g^{\alpha\beta}-g'^{\alpha\beta}\right)\frac{\partial}{\partial y_\beta}\left({}^tL_k\hat{u''}\right)+\left(\frac{g^{\alpha\beta}}{2g}\frac{\partial g}{\partial y_\alpha}-\frac{g'^{\alpha\beta}}{2g'}\frac{\partial g'}{\partial y_\alpha}\right)\frac{\partial}{\partial y_\beta}\left({}^tL_k\hat{u''}\right).$$

Note that

$$\frac{\partial g}{\partial y_\alpha}=\frac{\partial g_{11}}{\partial y_\alpha}g_{22}+\frac{\partial g_{22}}{\partial y_\alpha}g_{11}-2g_{12}\frac{\partial g_{12}}{\partial y_\alpha},$$

thus we only need to estimate

$$\left\|\left(g^{\alpha\beta}-g'^{\alpha\beta}\right)\frac{\partial g_{\gamma\zeta}}{\partial y_\alpha}\frac{\partial}{\partial y_\beta}\hat{u''}\right\|_{W_2^{l,l/2}(Q_t^{(k)})},\qquad \left\|\frac{\partial}{\partial y_\alpha}\left(g_{\alpha\beta}-g'_{\alpha\beta}\right)\frac{\partial}{\partial y_\beta}\hat{u''}\right\|_{W_2^{l,l/2}(Q_t^{(k)})}.$$

Thanks to (4.9)–(4.12), we get by using Hölder's inequality that

$$\left\|\left(g^{\alpha\beta}-g'^{\alpha\beta}\right)\frac{\partial g_{\gamma\zeta}}{\partial y_\alpha}\frac{\partial}{\partial y_\beta}\hat{u''}\right\|_{W_2^{l,l/2}(Q_t^{(k)})}+\left\|\frac{\partial}{\partial y_\alpha}\left(g_{\alpha\beta}-g'_{\alpha\beta}\right)\frac{\partial}{\partial y_\beta}\hat{u''}\right\|_{W_2^{l,l/2}(Q_t^{(k)})}$$
$$\leqslant C(t,\delta)t^{1/2-l/2}\left\|u''\right\|_{W_2^{l+2,l/2+1}(Q_t^{(k)})}\left\|u-u'\right\|_{W_2^{l+2,l/2+1}(Q_t^{(k)})}.$$

Therefore, we obtain

$$\left\|\left[\frac{1}{\sqrt{g}}\frac{\partial}{\partial y_\alpha}\left(\sqrt{g}g^{\alpha\beta}\right)-\frac{1}{\sqrt{g'}}\frac{\partial}{\partial y_\alpha}\left(\sqrt{g'}g'^{\alpha\beta}\right)\right]\frac{\partial}{\partial y_\beta}\left({}^tL_k\hat{u''}\right)\right\|_{W_2^{l,l/2}(Q_t^{(k)})}$$
$$\leqslant C(t,\delta)t^{1/2-l/2}\left\|u''\right\|_{W_2^{l+2,l/2+1}(Q_t^{(k)})}\left\|u-u'\right\|_{W_2^{l+2,l/2+1}(Q_t^{(k)})},$$

which together with (4.13) gives

$$\left\|\left(\Delta_u(t)-\Delta_{u'}(t)\right)u''\right\|_{W_2^{l,l/2}(Q_t^{(k)})}$$
$$\leqslant C(t,\delta)t^{1/2-l/2}\left\|u''\right\|_{W_2^{l+2,l/2+1}(Q_t^{(k)})}\left\|u-u'\right\|_{W_2^{l+2,l/2+1}(Q_t^{(k)})}. \tag{4.14}$$

Now, we estimate the other part of (i). A simple calculation gives

$$\dot{\Delta}_u(t)=-\frac{g_t}{2g}\Delta_u(t)+\frac{1}{\sqrt{g}}\frac{\partial}{\partial y_\alpha}\left[\left(\frac{\tilde{g}_{\alpha\beta}}{\sqrt{g}}\right)_t\frac{\partial}{\partial y_\beta}\right],$$

with $f_t\equiv\frac{\partial f}{\partial t}$ and

$$
\begin{aligned}
&\big(\dot{\Delta}_u(t) - \dot{\Delta}_{u'}(t)\big)\int_0^t u''\,d\tau \\
&= -\left(\frac{g_t}{2g}\Delta_u(t) - \frac{g'_t}{2g'}\Delta_{u'}(t)\right)\int_0^t {}^tL_k\hat{u''}\,d\tau \\
&\quad + \sum_{\alpha,\beta=1}^{2}\left\{\left[\frac{1}{\sqrt{g}}\left(\frac{\tilde{g}_{\alpha\beta}}{\sqrt{g}}\right)_t - \frac{1}{\sqrt{g'}}\left(\frac{\tilde{g}'_{\alpha\beta}}{\sqrt{g'}}\right)_t\right]\frac{\partial^2}{\partial y_\alpha \partial y_\beta}\right. \\
&\quad \left. + \left[\frac{1}{\sqrt{g}}\frac{\partial}{\partial y_\alpha}\left(\frac{\tilde{g}_{\alpha\beta}}{\sqrt{g}}\right)_t - \frac{1}{\sqrt{g'}}\frac{\partial}{\partial y_\alpha}\left(\frac{\tilde{g'}_{\alpha\beta}}{\sqrt{g'}}\right)_t\right]\frac{\partial}{\partial y_\beta}\right\}\int_0^t {}^tL_k\hat{u''}\,d\tau,
\end{aligned}
$$

from which and a similar proof of (4.14), we can infer that

$$
\begin{aligned}
&\left\|\big(\dot{\Delta}_u(t) - \dot{\Delta}_{u'}(t)\big)\int_0^t u''\,d\tau\right\|_{W_2^{l,l/2}(Q_t^{(k)})} \\
&\quad \leqslant C(t,\delta)t^{1/2-l/2}\left\|u-u'\right\|_{W_2^{l+2,l/2+1}(Q_t)}\left\|u''\right\|_{W_2^{l+2,l/2+1}(Q_t^{(k)})}.
\end{aligned}
$$

This proves (i). Since the proof of (ii) and (iii) is very similar, here we omit it. □

Lemma 4.4. *Assume that $u, u' \in W_2^{l+2,l/2+1}(Q_T)$ satisfying (4.4). Then there hold for $t \leqslant T$,*

$$
\begin{aligned}
&\text{(i)} \quad \left\|f(X_u,t)\right\|_{L^2(Q_t)} \leqslant t^{1/2}\|f\|_{L^\infty(0,t;L^2(\mathbb{R}^3))}; \\
&\text{(ii)} \quad \left\|f(X_u,t)\right\|_{W_2^{l,l/2}(Q_t)} \leqslant C(t,\delta)\|f\|_{W_2^{l+2,l/2+1}(\mathbb{R}^3_\infty)} \\
&\qquad\qquad + Ct^{3/2-l/2}\|f\|_{W_2^{l+3,l/2+3/2}(\mathbb{R}^3_\infty)}\|u\|_{W_2^{l+2,l/2+1}(Q_t)}; \\
&\text{(iii)} \quad \left\|f(X_u,t) - f(X_{u'},t)\right\|_{L^2(Q_t)} \leqslant t\|f\|_{W_2^{l+4,l/2+2}(\mathbb{R}^3_\infty)}\left\|u-u'\right\|_{L^2(Q_t)}; \\
&\text{(iv)} \quad \left\|f(X_u,t) - f(X_{u'},t)\right\|_{W_2^{l,l/2}(Q_t)} \leqslant C(t,\delta)t^{1/2}\|f\|_{W_2^{l+4,l/2+2}(\mathbb{R}^3_\infty)}\left\|u-u'\right\|_{W_2^{l+1,0}(Q_t)}.
\end{aligned}
$$

Proof. We only present the proof of (ii), the others can be similarly proved. (ii) can be deduced from the following inequalities

$$
\int_0^t \left\|f(X_u,\tau)\right\|_{L^2(\Omega)}^2 d\tau \leqslant \int_0^t \left\|f(\cdot,\tau)\right\|_{L^2(\mathbb{R}^3)}^2 d\tau \leqslant t\|f\|_{L^\infty(0,t;L^2(\mathbb{R}^3))}^2,
$$

$$
\begin{aligned}
&\int_0^t \left\|f(X_u,\tau)\right\|_{\dot{W}_2^l(\Omega)}^2 d\tau \\
&\quad = \int_0^t\int_\Omega\int_\Omega \frac{|f(X_u(\xi,\tau),\tau) - f(X_u(\tilde{\xi},\tau),\tau)|^2}{|\xi-\tilde{\xi}|^{3+2l}}\,d\xi\,d\tilde{\xi}\,d\tau
\end{aligned}
$$

$$= \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|f(x,\tau) - f(\tilde{x},\tau)|^2}{|x-\tilde{x}|^{3+2l}} \left(\frac{|x-\tilde{x}|}{|\xi(x) - \xi(\tilde{x})|} \right)^{3+2l} dx\, d\tilde{x}\, d\tau$$

$$\leqslant \int_0^t \|f(\cdot,\tau)\|^2_{\dot{W}^l_2(\mathbb{R}^3)} \left\| \int_0^\tau \frac{|u(\xi,s) - u(\tilde{\xi},s)|}{|\xi - \tilde{\xi}|} ds \right\|^{3+2l}_{L^\infty(\Omega\times\Omega)} d\tau$$

$$\leqslant C\big(t^{1/2} \|u\|_{W_2^{l+2,0}(Q_t)}\big)^{3+2l} \int_0^t \|f(\cdot,\tau)\|^2_{\dot{W}^l_2(\mathbb{R}^3)}\, d\tau,$$

$$\int_\Omega \|f(X_u,\tau)\|^2_{\dot{W}^{l/2}_2(0,t)}\, dx$$

$$= 2\int_\Omega \int_0^t \int_0^s \frac{|f(\xi + \int_0^s u\, dr, s) - f(\xi + \int_0^{s-\tau} u\, dr, s-\tau)|^2}{\tau^{1+l}}\, d\tau\, ds\, dx$$

$$\leqslant 4\int_\Omega \int_0^t \int_0^s \frac{|f(\xi + \int_0^s u\, dr, s) - f(\xi + \int_0^{s-\tau} u\, dr, s)|^2}{\tau^{1+l}}\, d\tau\, ds\, dx$$

$$+ 4\int_\Omega \int_0^t \int_0^s \frac{|f(\xi + \int_0^{s-\tau} u\, dr, s) - f(\xi + \int_0^{s-\tau} u\, dr, s-\tau)|^2}{\tau^{1+l}}\, d\tau\, ds\, dx$$

$$\leqslant C \sup_{\mathbb{R}^3_t} \big|\nabla f(x,s)\big|^2 \int_\Omega \int_0^t \int_0^s \frac{|\int_{s-\tau}^s u\, dr|^2}{\tau^{1+l}}\, d\tau\, ds\, dx + C \int_{\mathbb{R}^3} \|f(x,\cdot)\|^2_{\dot{W}^{l/2}_2(0,t)}\, dx$$

$$\leqslant C \sup_{\mathbb{R}^3_t} \big|\nabla f(x,s)\big|^2 \int_\Omega \|u(x,\cdot)\|^2_{L^\infty(0,t)}\, dx \int_0^t \int_0^s \tau^{1-l}\, d\tau\, ds + C\|f\|^2_{W_2^{0,l/2}(\mathbb{R}^3_t)}$$

$$\leqslant Ct^{3-l} \sup_{\mathbb{R}^3_t} \big|\nabla f(x,s)\big|^2 \|u\|^2_{W_2^{0,l/2+1/2}(Q_t)} + C\|f\|^2_{W_2^{0,l/2}(\mathbb{R}^3_t)}. \qquad \square$$

4.2. Approximate solutions and uniform estimates

We will solve (4.2)–(4.3) by successive approximations. Let $(\tilde{u}^{(1)(0)}, \tilde{q}^{(1)(0)}, \tilde{u}^{(2)(0)}, \tilde{q}^{(2)(0)}) = 0$ and define $(\tilde{u}^{(1)(m+1)}, \tilde{q}^{(1)(m+1)}, \tilde{u}^{(2)(m+1)}, \tilde{q}^{(2)(m+1)})$ $(m = 0, 1, 2, \ldots)$ as the solution to the linear problem:

$$\begin{cases} \rho_0^{(j)} \tilde{u}_t^{(j)(m+1)} - \nu^{(j)} \nabla^2 \tilde{u}^{(j)(m+1)} + \nabla \tilde{q}^{(j)(m+1)} \\ \quad = \nu^{(j)} \big({\nabla_m^{(j)}}^2 - \nabla^2\big) u^{(j)(m)} - \big(\nabla_m^{(j)} - \nabla\big) q^{(j)(m)} + \rho_0^{(j)} f^{(j)}(X_{u^{(j)(m)}}, t) \\ \quad \stackrel{\text{def}}{=} f^{(j)(m)} \quad \text{in } Q_T^{(j)}, \\ \nabla \cdot \tilde{u}^{(j)(m+1)} = \big(\nabla - \nabla_m^{(j)}\big) \cdot u^{(j)(m)} \stackrel{\text{def}}{=} g^{(j)(m)} \quad \text{in } Q_T^{(j)}, \\ \tilde{u}^{(j)(m+1)}\big|_{t=0} = v_0^{(j)} \quad \text{on } \Omega^{(j)},\ j = 1, 2, \end{cases} \tag{4.15}$$

together with the boundary conditions

$$\begin{cases} \Pi^{(1)}\big[\nu^{(1)}\mathrm{D}\big(\tilde u^{(1)(m+1)}\big)N^{(1)}\big]\big|_{S^{(1)}_{F,T}} \\ \quad = \nu^{(1)}\Pi^{(1)}\big[\mathrm{D}\big(u^{(1)(m)}\big)N^{(1)} - \Pi^{(1)}_{u^{(1)(m)}}\big[\mathrm{D}^{(1)}_m\big(u^{(1)(m)}\big)n^{(1)}_{u^{(1)(m)}}\big]\big]\big|_{S^{(1)}_{F,T}} \overset{\mathrm{def}}{=} \vec d^{(1)(m)}, \\ -\tilde q^{(1)(m+1)} + 2\nu^{(1)}N^{(1)}\cdot\big(\nabla\tilde u^{(1)(m+1)}N^{(1)}\big) - \sigma^{(1)}N^{(1)}\cdot\Delta^{(1)}\displaystyle\int_0^t \tilde u^{(1)(m+1)}\,d\tau\Big|_{S^{(1)}_{F,T}} \\ \quad = b^{(1)(m)} + \sigma^{(1)}\displaystyle\int_0^t B^{(1)(m)}\,d\tau - g_0\int_0^t u_3^{(1)(m)}\,d\tau\Big|_{S^{(1)}_{F,T}}, \\ \tilde u^{(1)(m+1)} - \tilde u^{(2)(m+1)}\big|_{S^{(2)}_{F,T}} = 0, \\ \Pi^{(2)}\big[\nu^{(2)}\mathrm{D}\big(\tilde u^{(2)(m)}\big)N^{(2)} - \nu^{(1)}\mathrm{D}\big(\tilde u^{(1)(m)}\big)N^{(2)}\big]\big|_{S^{(2)}_{F,T}} \\ \quad = \Pi^{(2)}\Big[\displaystyle\sum_{j=1}^{2}(-1)^j\nu^{(j)}\big(\mathrm{D}\big(u^{(j)(m)}\big)N^{(2)} - \Pi^{(2)}_{u^{(j)(m)}}\big[\mathrm{D}^{(j)}_m\big(u^{(j)(m)}\big)n^{(2)}_{u^{(j)(m)}}\big]\big)\Big]\Big|_{S^{(2)}_{F,T}} \\ \quad \overset{\mathrm{def}}{=} \vec d^{(2)(m)}, \\ \tilde q^{(1)(m+1)} - \tilde q^{(2)(m+1)} + 2\nu^{(2)}N^{(2)}\cdot\big(\nabla\tilde u^{(2)(m+1)}N^{(2)}\big) \\ \qquad - 2\nu^{(1)}N^{(2)}\cdot\big(\nabla\tilde u^{(1)(m+1)}N^{(2)}\big) - \dfrac{\sigma^{(2)}}{2}N^{(2)}\cdot\Delta^{(2)}\displaystyle\int_0^t\sum_{j=1}^{2}\tilde u^{(j)(m+1)}\,d\tau\Big|_{S^{(2)}_{F,T}} \\ \quad = b^{(2)(m)} + \dfrac{\sigma^{(2)}}{2}\displaystyle\int_0^t B^{(2)(m)}\,d\tau - g_0\int_0^t u_3^{(2)(m)}\,d\tau\Big|_{S^{(2)}_{F,T}}, \\ \tilde u^{(2)(m+1)}\big|_{S^{(2)}_{B,T}} = 0, \end{cases} \tag{4.16}$$

provided that $(u^{(j)(m)}, q^{(j)(m)}) = (\bar u^{(j)} + \tilde u^{(j)(m)}, \bar q^{(j)} + \tilde q^{(j)(m)})$ given in such a way that $(u^{(j)(m)}, q^{(j)(m)}, \nabla q^{(j)(m)}) \in W_2^{l+2,l/2+1}(Q_T^{(j)}) \times W_2^{l,l/2}(Q_T^{(j)}) \times W_2^{l,l/2}(Q_T^{(j)})$ and

$$T^{1/2-l/2}\big\|u^{(j)(m)}\big\|_{W_2^{l+2,l/2+1}(Q_T^{(j)})} \leqslant \delta, \quad j=1,2, \tag{4.17}$$

with small enough positive constant δ. Here $0 < T < +\infty$, $\nabla^{(j)}_m = \nabla_{u^{(j)(m)}} = \mathcal{A}_{u^{(j)(m)}}\nabla$, $\mathrm{D}^{(j)}_m = \mathrm{D}_{u^{(j)(m)}}$, $b^{(1)(m)} = b^{(1)}(u^{(1)(m)})$, $b^{(2)(m)} = b^{(2)}(u^{(1)(m)}, u^{(2)(m)})$, $B^{(1)(m)} = B^{(1)}(u^{(1)(m)})$, $B^{(2)(m)} = B^{(2)}(u^{(1)(m)}, u^{(2)(m)})$ and $g^{(j)(m)} = \nabla\cdot R^{(j)(m)}$ with

$$R^{(j)(m)} = \big(I - {}^t\mathcal{A}_{u^{(j)(m)}}\big)u^{(j)(m)}, \quad j=1,2.$$

Theorem 3.1 applied ensures that

$$\begin{aligned}\tilde Y^{(m+1)}(t) \overset{\mathrm{def}}{=} &\sum_{j=1}^{2}\big(\big\|\tilde u^{(j)(m+1)}\big\|_{W_2^{l+2,l/2+1}(Q_t^{(j)})} + \big\|\nabla\tilde q^{(j)(m+1)}\big\|_{W_2^{l,l/2}(Q_t^{(j)})} + \big\|\tilde q^{(j)(m+1)}\big\|_{W_2^{l,l/2}(Q_t^{(j)})}\big) \\ &+ \big\|\tilde q^{(1)(m+1)}\big\|_{W_2^{l+1/2,l/2+1/4}(S^{(1)}_{F,t})} + \big\|\tilde q^{(1)(m+1)} - \tilde q^{(2)(m+1)}\big\|_{W_2^{l+1/2,l/2+1/4}(S^{(2)}_{F,t})}\end{aligned}$$

$$\begin{aligned}
&\leqslant C(t)\sum_{j=1}^{2}\Bigg\{\big\|v_0^{(j)}\big\|_{W_2^{l+1}(\Omega^{(j)})}+\big\|f^{(j)(m)}\big\|_{W_2^{l,l/2}(Q_t^{(j)})}+\big\|g^{(j)(m)}\big\|_{W_2^{l+1,l/2+1/2}(Q_t^{(j)})}\\
&\quad+\big\|R^{(j)(m)}\big\|_{W_2^{0,l/2+1}(Q_t^{(j)})}+\big\|\vec{d}^{(j)(m)}\big\|_{W_2^{l+1/2,l/2+1/4}(S_{F,t}^{(j)})}+\big\|b^{(j)(m)}\big\|_{W_2^{l+1/2,l/2+1/4}(S_{F,t}^{(j)})}\\
&\quad+\big\|F_0^{(j)}-h^{(j)}\big\|_{W_2^{l+1/2}(\mathbb{R}^2)}+\bigg\|\int_0^t u_3^{(j)(m)}\,d\tau\bigg\|_{W_2^{l+1/2,l/2+1/4}(S_{F,t}^{(j)})}\\
&\quad+\sigma^{(j)}\big\|B^{(j)(m)}\big\|_{W_2^{l-1/2,l/2-1/4}(S_{F,t}^{(j)})}\Bigg\}. \qquad (4.18)
\end{aligned}$$

Now, we estimate the terms on the right-hand side of (4.18) by using Lemmas 4.1–4.4 with $u=u^{(j)(m)}$ and $u'=0$.

Step 1. The estimate of $f^{(j)(m)}$. It is easy to get that

$$\begin{aligned}
&\big\|\big(\nabla_m^{(j)}-\nabla\big)q^{(j)(m)}\big\|_{W_2^{l,l/2}(Q_t^{(j)})}\\
&\quad=\big\|(\mathcal{A}_{u^{(j)(m)}}-I)\nabla q^{(j)(m)}\big\|_{W_2^{l,l/2}(Q_t^{(j)})}\\
&\quad\leqslant C\big(\|\mathcal{A}_{u^{(j)(m)}}-I\|_{L^\infty(0,t;W_2^{l+1}(\Omega^{(j)}))}+\|\mathcal{A}_{u^{(j)(m)}}-I\|_{L^\infty(\Omega^{(j)};W_2^{l/2+1/2}(0,t))}\big)\\
&\qquad\times\big\|\nabla q^{(j)(m)}\big\|_{W_2^{l,l/2}(Q_t^{(j)})}\\
&\quad\leqslant C(t,\delta)t^{1/2-l/2}\big\|u^{(j)(m)}\big\|_{W_2^{l+2,l/2+1}(Q_t^{(j)})}\big\|\nabla q^{(j)(m)}\big\|_{W_2^{l,l/2}(Q_t^{(j)})},
\end{aligned}$$

and similarly

$$\begin{aligned}
&\big\|\big({\nabla_m^{(j)}}^2-\nabla^2\big)u^{(j)(m)}\big\|_{W_2^{l,l/2}(Q_t^{(j)})}\\
&\quad\leqslant\big\|{}^t\mathcal{A}_{u^{(j)(m)}}(\mathcal{A}_{u^{(j)(m)}}-I)\nabla\cdot\nabla u^{(j)(m)}\big\|_{W_2^{l,l/2}(Q_t^{(j)})}\\
&\qquad+\big\|\big[(\mathcal{A}_{u^{(j)(m)}}-I)\nabla\cdot\mathcal{A}_{u^{(j)(m)}}\big]\cdot\nabla u^{(j)(m)}\big\|_{W_2^{l,l/2}(Q_t^{(j)})}\\
&\qquad+\big\|{}^t(\mathcal{A}_{u^{(j)(m)}}-I)\nabla\cdot\nabla u^{(j)(m)}\big\|_{W_2^{l,l/2}(Q_t^{(j)})}+\big\|(\nabla\cdot\mathcal{A}_{u^{(j)(m)}})\cdot\nabla u^{(j)(m)}\big\|_{W_2^{l,l/2}(Q_t^{(j)})}\\
&\quad\leqslant C(t,\delta)t^{1/2-l/2}\big\|u^{(j)(m)}\big\|^2_{W_2^{l+2,l/2+1}(Q_t^{(j)})},
\end{aligned}$$

where we have also used Remark 4.1 in order to estimate the term $\|(\nabla\cdot\mathcal{A}_{u^{(j)(m)}})\cdot\nabla u^{(j)(m)}\|_{W_2^{l,l/2}(Q_t^{(j)})}$. We also have

$$\big\|\rho_0^{(j)}f^{(j)}(X_{u^{(j)(m)}},t)\big\|_{W_2^{l,l/2}(Q_t^{(j)})}\leqslant C(t,\delta)\big\|f^{(j)}\big\|_{W_2^{l+4,l/2+2}(\mathbb{R}^3_\infty)},$$

which together with the above two inequalities implies that

$$\big\|f^{(j)(m)}\big\|_{W_2^{l,l/2}(Q_t^{(j)})}\leqslant C(t,\delta)t^{1/2-l/2}Y^{(m)}(t)^2+C(t,\delta)\big\|f^{(j)}\big\|_{W_2^{l+4,l/2+2}(\mathbb{R}^3_\infty)},$$

where

$$Y^{(m)}(t) \overset{\text{def}}{=} \sum_{j=1}^{2}\big(\|u^{(j)(m)}\|_{W_2^{l+2,l/2+1}(Q_t^{(j)})} + \|\nabla q^{(j)(m)}\|_{W_2^{l,l/2}(Q_t^{(j)})} + \|q^{(j)(m)}\|_{W_2^{l,l/2}(Q_t^{(j)})}\big) + \|q^{(1)(m)}\|_{W_2^{l+1/2,l/2+1/4}(S_{F,t}^{(1)})} + \|q^{(1)(m)} - q^{(2)(m)}\|_{W_2^{l+1/2,l/2+1/4}(S_{F,t}^{(2)})}.$$

Step 2. The estimates of $g^{(j)(m)}$ and $R^{(j)(m)}$. Similar to the estimate of $f^{(j)(m)}$, we can obtain

$$\|g^{(j)(m)}\|_{W_2^{l+1,l/2+1/2}(Q_t^{(j)})} + \|R^{(j)(m)}\|_{W_2^{0,l/2+1}(Q_t^{(j)})} \leqslant C(t,\delta)t^{1/2-l/2}Y^{(m)}(t)^2.$$

Step 3. The estimates of $\vec{d}^{(1)(m)}$ and $\vec{d}^{(2)(m)}$. Notice that $\vec{d}^{(1)(m)} = \nu^{(1)}(\mathrm{I}+\mathrm{II})$ with

$$\begin{aligned}\mathrm{I} &= \mathrm{D}\big(u^{(1)(m)}\big)N^{(1)} - \mathrm{D}_m^{(1)}\big(u^{(1)(m)}\big)n^{(1)}_{u^{(1)(m)}}\\ &= (I-\mathcal{A}_{u^{(1)(m)}})\nabla u^{(1)(m)}N^{(1)} + \mathcal{A}_{u^{(1)(m)}}\nabla u^{(1)(m)}\big(N^{(1)} - n^{(1)}_{u^{(1)(m)}}\big)\\ &\quad + {}^t\big[(I-\mathcal{A}_{u^{(1)(m)}})\nabla u^{(1)(m)}N^{(1)} + \mathcal{A}_{u^{(1)(m)}}\nabla u^{(1)(m)}\big(N^{(1)} - n^{(1)}_{u^{(1)(m)}}\big)\big],\end{aligned}$$

and

$$\begin{aligned}\mathrm{II} &= \big(\mathrm{D}\big(u^{(1)(m)}\big)N^{(1)}\cdot N^{(1)}\big)N^{(1)} - \big(\mathrm{D}_m^{(1)}\big(u^{(1)(m)}\big)n^{(1)}_{u^{(1)(m)}}\cdot n^{(1)}_{u^{(1)(m)}}\big)\big(n^{(1)}_{u^{(1)(m)}}\cdot N^{(1)}\big)N^{(1)}\\ &= \big\{\big[\mathrm{D}\big(u^{(1)(m)}\big)N^{(1)} - \mathrm{D}_m^{(1)}\big(u^{(1)(m)}\big)n^{(1)}_{u^{(1)(m)}}\big]\cdot N^{(1)}\\ &\quad + \mathrm{D}_m^{(1)}\big(u^{(1)(m)}\big)n^{(1)}_{u^{(1)(m)}}\cdot\big(N^{(1)} - n^{(1)}_{u^{(1)(m)}}\big)\\ &\quad + \big[\mathrm{D}_m^{(1)}\big(u^{(1)(m)}\big)n^{(1)}_{u^{(1)(m)}}\cdot n^{(1)}_{u^{(1)(m)}}\big]\big(\big(N^{(1)} - n^{(1)}_{u^{(1)(m)}}\big)\cdot N^{(1)}\big)\big\}N^{(1)},\end{aligned}$$

from which, we can get

$$\|\vec{d}^{(1)(m)}\|_{W_2^{l+1/2,l/2+1/4}(S_{F,t}^{(1)})} \leqslant C(t,\delta)t^{1/2-l/2}Y^{(m)}(t)^2.$$

The same estimate holds for $\vec{d}^{(2)(m)}$.

Step 4. The estimate of $b^{(1)}(u^{(1)(m)})$ and $b^{(2)}(u^{(1)(m)}, u^{(2)(m)})$. Notice that

$$H_0^{(1)}(\xi) = \nabla'_\xi\cdot\frac{\nabla'_\xi F_0^{(1)}}{\sqrt{1+|\nabla'_\xi F_0^{(1)}|^2}},$$

whence

$$\|H_0^{(1)}(\xi)\|_{W_2^{l+1/2,l/2+1/4}(S_{F,t}^{(1)})} \leqslant Ct^{1/2}\|F_0^{(1)} - h^{(1)}\|_{W_2^{l+5/2}(\mathbb{R}^2)},$$

which together with the definition of $b^{(1)}(u^{(1)(m)})$, we can get

$$\begin{aligned}\big\|b^{(1)}\big(u^{(1)(m)}\big)\big\|_{W_2^{l+1/2,l/2+1/4}(S_{F,t}^{(1)})} &\leqslant C(t,\delta)t^{1/2-l/2}Y^{(m)}(t)^2\\ &\quad + Ct^{1/2}\sigma^{(1)}\big\|F_0^{(1)}-h^{(1)}\big\|_{W_2^{l+5/2}(\mathbb{R}^2)}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\big\|b^{(2)}\big(u^{(1)(m)},u^{(2)(m)}\big)\big\|_{W_2^{l+1/2,l/2+1/4}(S_{F,t}^{(2)})} &\leqslant C(t,\delta)t^{1/2-l/2}Y^{(m)}(t)^2\\ &\quad + Ct^{1/2}\sigma^{(2)}\big\|F_0^{(2)}-h^{(2)}\big\|_{W_2^{l+5/2}(\mathbb{R}^2)}.\end{aligned}$$

Step 5. The estimates of $B^{(1)}(u^{(1)(m)})$ and $B^{(2)}(u^{(1)(m)},u^{(2)(m)})$.

First of all, for the term $(n_{u^{(1)(m)}}^{(1)}\cdot\Delta_{u^{(1)(m)}}^{(1)}-N^{(1)}\cdot\Delta_0^{(1)})u^{(1)(m)}$, we have

$$\begin{aligned}&\big\|\big(n_{u^{(1)(m)}}^{(1)}\cdot\Delta_{u^{(1)(m)}}^{(1)}-N^{(1)}\cdot\Delta_0^{(1)}\big)u^{(1)(m)}\big\|_{W_2^{l-1/2,l/2-1/4}(S_{F,t}^{(1)})}\\ &\quad\leqslant\big\|\big(n_{u^{(1)(m)}}^{(1)}-N^{(1)}\big)\cdot\Delta_{u^{(1)(m)}}^{(1)}u^{(1)(m)}\big\|_{W_2^{l-1/2,l/2-1/4}(S_{F,t}^{(1)})}\\ &\qquad+\big\|N^{(1)}\cdot\big(\Delta_{u^{(1)(m)}}^{(1)}-\Delta_0^{(1)}\big)u^{(1)(m)}\big\|_{W_2^{l-1/2,l/2-1/4}(S_{F,t}^{(1)})}\\ &\quad\leqslant C(t,\delta)t^{1/2-l/2}Y^{(m)}(t)^2.\end{aligned}$$

For the term

$$\frac{\partial}{\partial t}n_{u^{(1)(m)}}^{(1)}\cdot\Delta_{u^{(1)(m)}}^{(1)}\xi+n_{u^{(1)(m)}}^{(1)}\cdot\dot{\Delta}_{u^{(1)(m)}}^{(1)}\xi,$$

we represent them in the Cartesian coordinates system $\{y\}$ on $S_F^{(k'')}$. We know

$$\Delta_{u^{(1)(m)}}^{(1)}\xi=g^{\alpha\beta}\frac{\partial^2}{\partial y_\alpha\partial y_\beta}\xi+\frac{1}{\sqrt{g}}\frac{\partial}{\partial y_\alpha}\big(\sqrt{g}g^{\alpha\beta}\big)\frac{\partial}{\partial y_\beta}\xi,$$

where $\xi=\xi^{(k'')}+{}^tL_{k''}y$ with $y_3=\varphi^{(1)}(y_1,y_2)$ on $S_F^{(k'')}$, and $g_{\alpha\beta},g^{\alpha\beta}$ are defined as that in the proof of Lemma 4.3 with $u=u^{(1)(m)}$. So on $S_F^{(k'')}$,

$$L_{k''}\frac{\partial}{\partial y_1}\xi={}^t\big(1,0,\varphi_{y_1}^{(1)}\big),\qquad L_{k''}\frac{\partial}{\partial y_2}\xi={}^t\big(1,0,\varphi_{y_2}^{(1)}\big),\qquad L_{k''}\frac{\partial^2}{\partial y_\alpha\partial y_\beta}\xi={}^t\big(0,0,\varphi_{y_\alpha y_\beta}^{(1)}\big)$$

from which, we get by extending all functions from $S_F^{(k'')}$ to $\Omega_{k''}$ that

$$\begin{aligned}&\left\|\frac{\partial}{\partial t}n_{u^{(1)(m)}}^{(1)}\cdot\Delta_{u^{(1)(m)}}^{(1)}\xi\right\|_{W_2^{l,l/2}(Q_t^{(k'')})}\\ &\quad\leqslant C\big\|\nabla\nabla\varphi^{(1)}\big\|_{W_2^{l+1}(\mathbb{R}_+^3)}\left\|\frac{\partial}{\partial t}n_{u^{(1)(m)}}^{(1)}g^{\alpha\beta}\right\|_{W_2^{l,l/2}(Q_t^{(k'')})}\\ &\qquad+C\big(1+\big\|\nabla\varphi^{(1)}\big\|_{W_2^{l+1}(\mathbb{R}_+^3)}\big)\left\|\frac{\partial}{\partial t}n_{u^{(1)(m)}}^{(1)}\frac{1}{\sqrt{g}}\frac{\partial}{\partial y_\alpha}\big(\sqrt{g}g^{\alpha\beta}\big)\right\|_{W_2^{l,l/2}(Q_t^{(k'')})}\end{aligned}$$

$$\leqslant C\left\|\frac{\partial}{\partial t}n^{(1)}_{u^{(1)(m)}}g^{\alpha\beta}\right\|_{W_2^{l,l/2}(Q_t^{(k'')})}+C(t,\delta)\left\|\frac{\partial}{\partial t}n^{(1)}_{u^{(1)(m)}}\frac{\partial g^{\alpha\beta}}{\partial y_\alpha}\right\|_{W_2^{l,l/2}(Q_t^{(k'')})}.$$

From the formula

$$\begin{aligned}\frac{\partial}{\partial t}n^{(1)}_{u^{(1)(m)}}=\operatorname{Det}\left(\frac{\partial X_{u^{(j)}}}{\partial\xi}\right)^{-1}\big[-\mathcal{A}_{u^{(1)(m)}}{}^t\big(\nabla u^{(1)(m)}\big)n^{(1)}_{u^{(1)(m)}}\\-n^{(1)}_{u^{(1)(m)}}\big(n^{(1)}_{u^{(1)(m)}}\cdot\mathcal{A}_{u^{(1)(m)}}{}^t\big(\nabla u^{(1)(m)}\big)n^{(1)}_{u^{(1)(m)}}\big)\big],\end{aligned}\tag{4.19}$$

we get

$$\left\|\frac{\partial}{\partial t}n^{(1)}_{u^{(1)(m)}}g^{\alpha\beta}\right\|_{W_2^{l,l/2}(Q_t^{(k'')})}\leqslant C(t,\delta)\left\|\nabla u^{(1)(m)}\right\|_{W_2^{l,l/2}(Q_t^{(k'')})}.\tag{4.20}$$

Notice that $\|\nabla u\|^2_{W_2^l(\Omega)}=\|\nabla u\|^2_{L^2(\Omega)}+\|\nabla u\|^2_{\dot W_2^l(\Omega)}$ and the interpolation inequalities

$$\begin{aligned}\|\nabla u\|_{L^2(\Omega)}&\leqslant\epsilon\|\nabla u\|_{\dot W_2^{l+1}(\Omega)}+C\epsilon^{-\frac{1}{l+1}}\|u\|_{L^2(\Omega)},\\\|\nabla u\|_{\dot W_2^l(\Omega)}&\leqslant\epsilon\|\nabla u\|_{\dot W_2^{l+1}(\Omega)}+C\epsilon^{-l-1}\|u\|_{L^2(\Omega)},\\\|\nabla u\|_{\dot W_2^{l/2}(0,t)}&\leqslant\epsilon\|\nabla u\|_{\dot W_2^{l/2+1/2}(0,t)}+\epsilon^{-l}\|\nabla u\|_{L^2(0,t)}.\end{aligned}\tag{4.21}$$

We take $\epsilon=t^{\frac{1}{2(l+2)}}$ in (4.21), and use the fact that

$$\begin{aligned}\|u\|_{L^2(Q_t)}&\leqslant t^{1/2}\|u\|_{L^2(\Omega;L^\infty(0,t))}\leqslant Ct^{1/2}\|u\|_{W_2^{l+2,l/2+1}(Q_t)},\\\|\nabla u\|_{L^2(Q_t)}&\leqslant t^{1/2}\|\nabla u\|_{L^2(\Omega;L^\infty(0,t))}\leqslant Ct^{1/2}\|u\|_{W_2^{l+2,l/2+1}(Q_t)},\end{aligned}$$

then we obtain

$$\left\|\nabla u^{(1)(m)}\right\|_{W_2^{l,l/2}(Q_t^{(k'')})}\leqslant C(t,\delta)t^{\frac{1}{2(l+2)}}\left\|u^{(1)(m)}\right\|_{W_2^{l+2,l/2+1}(Q_t^{(k'')})},\tag{4.22}$$

from which and (4.20), we obtain

$$\left\|\frac{\partial}{\partial t}n^{(1)}_{u^{(1)(m)}}g^{\alpha\beta}\right\|_{W_2^{l,l/2}(Q_t^{(k'')})}\leqslant C(t,\delta)t^{\frac{1}{2(l+2)}}\left\|u^{(1)(m)}\right\|_{W_2^{l+2,l/2+1}(Q_t^{(k'')})}.\tag{4.23}$$

On the other hand, we get by (4.19) that

$$\left\|\frac{\partial}{\partial t}n^{(1)}_{u^{(1)(m)}}\frac{\partial g^{\alpha\beta}}{\partial y_\alpha}\right\|_{W_2^l(\Omega_{k''})}\leqslant C(t,\delta)\left\|\nabla u^{(1)(m)}\right\|_{W_2^{l+1-2\theta}(\Omega_{k''})},$$

with $\theta\in(0,l/2-1/4)$ and

$$\begin{aligned}
&\left\| \frac{\partial}{\partial t} n^{(1)}_{u^{(1)(m)}} \frac{\partial g^{\alpha\beta}}{\partial y_\alpha} \right\|_{W_2^{l/2}(0,t)} \\
&\quad \leqslant C \left\| \frac{\partial}{\partial t} n^{(1)}_{u^{(1)(m)}} \right\|_{W_2^{l/2}(0,t)} \left\| \nabla g^{\alpha\beta} \right\|_{W_2^{l/2+1/2}(0,t)} \\
&\quad \leqslant C(t,\delta) t^{1/2-l/2} \left\| \nabla u^{(1)(m)} \right\|_{W_2^{l/2}(0,t)} \left(\left\| \nabla\nabla u^{(1)(m)} \right\|_{L^2(0,t)} + \left\| \nabla u^{(1)(m)} \right\|_{L^2(0,t)} \right),
\end{aligned}$$

where we have used (4.10) and the definition of $g^{\alpha\beta}$ in the last inequality.

Therefore, we get by using Hölder's inequality and the similar interpolation inequalities as (4.21) that

$$\left\| \frac{\partial}{\partial t} n^{(1)}_{u^{(1)(m)}} \frac{\partial g^{\alpha\beta}}{\partial y_\alpha} \right\|_{W_2^{l,l/2}(Q_t^{(k'')})} \leqslant C(t,\delta) t^{\frac{\theta}{l+2}} \left\| u^{(1)(m)} \right\|^2_{W_2^{l+2,l/2+1}(Q_t^{(k'')})},$$

which together with (4.23) and the trace theorem implies that

$$\left\| \frac{\partial}{\partial t} n^{(1)}_{u^{(1)(m)}} \cdot \Delta^{(1)}_{u^{(1)(m)}} \xi \right\|_{W_2^{l-1/2,l/2-1/4}(S^{(1)}_{F,t})} \leqslant C(t,\delta) t^{\frac{\theta}{l+2}} Y^{(m)}(t) \left(Y^{(m)}(t) + 1 \right). \tag{4.24}$$

Now we turn to consider

$$n^{(1)}_{u^{(1)(m)}} \cdot \dot{\Delta}^{(1)}_{u^{(1)(m)}} \xi = \left(n^{(1)}_{u^{(1)(m)}} - N^{(1)} \right) \cdot \dot{\Delta}^{(1)}_{u^{(1)(m)}} \xi + N^{(1)} \cdot \dot{\Delta}^{(1)}_{u^{(1)(m)}} \xi,$$

since

$$\dot{\Delta}^{(1)}_{u^{(1)(m)}} \xi = -\frac{g_t}{2g} \Delta^{(1)}_{u^{(1)(m)}} \xi + \frac{1}{\sqrt{g}} \frac{\partial}{\partial y_\alpha} \left[\left(\sqrt{g} g^{\alpha\beta} \right)_t \frac{\partial}{\partial y_\beta} \xi \right],$$

we have

$$\left\| \left(n^{(1)}_{u^{(1)(m)}} - N^{(1)} \right) \cdot \dot{\Delta}^{(1)}_{u^{(1)(m)}} \xi \right\|_{W_2^{l,l/2}(Q_t^{(k'')})} \leqslant C(t,\delta) t^{1/2-l/2} \left\| u^{(1)(m)} \right\|^2_{W_2^{l+2,l/2+1}(Q_t^{(k'')})}. \tag{4.25}$$

Thanks to

$$N^{(1)} = {}^t\left(\frac{\varphi^{(1)}_{z_1}}{\sqrt{1+|\nabla'_z \varphi^{(1)}|^2}}, \frac{\varphi^{(1)}_{z_2}}{\sqrt{1+|\nabla'_z \varphi^{(1)}|^2}}, \frac{-1}{\sqrt{1+|\nabla'_z \varphi^{(1)}|^2}} \right)$$

in the local coordinate $\{y_1, y_2\} = \{z_1, z_2\}$, where $y_1 = z_1$, $y_2 = z_2$, $y_3 = z_3 + \varphi^{(1)}(z')$ with $z_3 = 0$ representing $S_F^{(k'')}$, we find that

$$N^{(1)} \cdot \dot{\Delta}^{(1)}_{u^{(1)(m)}} \xi = \left(1 + \left| \nabla_{z'} \varphi^{(1)} \right|^2 \right)^{-1/2} \frac{1}{\sqrt{g}} \left(\frac{g_t}{2g^{3/2}} \tilde{g}_{\alpha\beta} - \left(\frac{\tilde{g}_{\alpha\beta}}{\sqrt{g}} \right)_t \right) \frac{\partial^2}{\partial z_\alpha \partial z_\beta} \varphi^{(1)}(z_1, z_2),$$

and notice that $\frac{g_t}{2g^{3/2}} \tilde{g}_{\alpha\beta} - (\frac{\tilde{g}_{\alpha\beta}}{\sqrt{g}})_t = \frac{g_t}{g^{3/2}} \tilde{g}_{\alpha\beta} - \frac{1}{\sqrt{g}} \frac{\partial \tilde{g}_{\alpha\beta}}{\partial t}$, we obtain

$$\begin{aligned}\big\|N^{(1)}\cdot\dot{\Delta}^{(1)}_{u^{(1)(m)}}\xi\big\|_{W_2^{l,l/2}(Q_t^{(k'')})} &\leqslant C(t,\delta)\left\|g_{\gamma\zeta}\frac{\partial\tilde{g}_{\alpha\beta}}{\partial t}\right\|_{W_2^{l,l/2}(Q_t^{(k'')})}\\ &\leqslant C(t,\delta)\big\|\nabla u^{(1)(m)}\big\|_{W_2^{l,l/2}(Q_t^{(k'')})}\\ &\leqslant C(t,\delta)t^{\frac{1}{2(l+2)}}\big\|u^{(1)(m)}\big\|_{W_2^{l+2,l/2+1}(Q_t^{(k'')})},\end{aligned} \tag{4.26}$$

where we have used the expression of $g_{\alpha\beta}$ in the second inequality and used (4.22) in the last inequality. Thanks to (4.24)–(4.26), we get by the trace theorem that

$$\begin{aligned}&\left\|\frac{\partial}{\partial t}n^{(1)}_{u^{(1)(m)}}\cdot\Delta^{(1)}_{u^{(1)(m)}}\xi\right\|_{W_2^{l-1/2,l/2-1/4}(S^{(1)}_{F,t})}+\big\|n^{(1)}_{u^{(1)(m)}}\cdot\dot{\Delta}^{(1)}_{u^{(1)(m)}}\xi\big\|_{W_2^{l-1/2,l/2-1/4}(S^{(1)}_{F,t})}\\ &\quad\leqslant C(t,\delta)t^{\frac{\theta}{l+2}}Y^{(m)}(t)\big(Y^{(m)}(t)+1\big).\end{aligned}$$

Similarly, we can obtain

$$\begin{aligned}&\left\|\frac{\partial}{\partial t}n^{(1)}_{u^{(1)(m)}}\cdot\Delta^{(1)}_{u^{(1)(m)}}\int_0^t u^{(1)(m)}\,d\tau+n^{(1)}_{u^{(1)(m)}}\cdot\dot{\Delta}^{(1)}_{u^{(1)(m)}}\int_0^t u^{(1)(m)}\,d\tau\right\|_{W_2^{l-1/2,l/2-1/4}(S^{(1)}_{F,t})}\\ &\quad\leqslant C(t,\delta)t^{1/2-l/2}Y^{(m)}(t)^2.\end{aligned}$$

Therefore, we arrive at

$$\big\|B^{(1)}\big(u^{(1)(m)}\big)\big\|_{W_2^{l-1/2,l/2-1/4}(S^{(1)}_{F,t})}\leqslant C(t,\delta)t^{\frac{\theta}{l+2}}Y^{(m)}(t)\big(Y^{(m)}(t)+1\big),$$

analogously, we have

$$\big\|B^{(2)}\big(u^{(1)(m)},u^{(2)(m)}\big)\big\|_{W_2^{l-1/2,l/2-1/4}(S^{(2)}_{F,t})}\leqslant C(t,\delta)t^{\frac{\theta}{l+2}}Y^{(m)}(t)\big(Y^{(m)}(t)+1\big).$$

Remark 4.2. In Step 5, to estimate $B^{(1)}(u^{(1)(m)}), B^{(2)}(u^{(1)(m)},u^{(2)(m)})$, we used that $S_F^{(j)}\in W_2^{l+5/2}$ i.e. $\varphi^{(j)}\in W_2^{l+3}(\mathbb{R}^3_+)$. In fact, by the more precise calculations, we only need that $S_F^{(j)}\in W_2^{l+3/2}$.

Combining Steps 1–5, we get by (4.18) that

$$\begin{aligned}\tilde{Y}^{(m+1)}(t)\leqslant C(t)\Bigg\{&\sum_{j=1}^{2}\Big(\big\|f^{(j)}\big\|_{W_2^{l+4,l/2+2}(\mathbb{R}^3_\infty)}+\big\|v_0^{(j)}\big\|_{W_2^{l+1}(\Omega^{(j)})}\\ &+\sigma^{(j)}\big\|F_0^{(j)}-h^{(j)}\big\|_{W_2^{l+5/2}(\mathbb{R}^2)}\Big)+C(t,\delta)t^{\frac{\theta}{l+2}}Y^{(m)}(t)\big(Y^{(m)}(t)+1\big)\Bigg\}.\end{aligned}$$

Then, thanks to (4.1), we can get

$$\tilde{Y}^{(m+1)}(t)\leqslant a_0+a_1\tilde{Y}^{(m)}(t)+a_2\tilde{Y}^{(m)}(t)^2.$$

Here a_0 is a positive constant depending on t increasingly, and a_1,a_2 are positive constants depending on t,δ increasingly. Furthermore, $a_1,a_2\to 0$ as $t\to 0$ and as $\delta\to 0$.

We take T small enough such that

$$a_1 \leqslant 1/2, \qquad a_2 \leqslant \frac{1}{32a_0}, \qquad (a_1-1)^2 - 4a_0a_2 \in (1/8, 1).$$

Hence, the equation $a_2Y^2 + (a_1-1)Y + a_0 = 0$ has roots and the smaller root at $t = T_0$ denoted by

$$Y^* = \frac{1 - a_1 - \sqrt{(1-a_1)^2 - 4a_0a_2}}{2a_2}\bigg|_{t=T_0} \in \big[a_0(T_0), 4a_0(T_0)\big]$$

such that (4.4) holds for $T = T_0$.

Therefore, $\tilde{Y}^{(m)}(t) \leqslant Y^*$ for all $m \geqslant 0$ and for any $t \in (0, T_0)$.

Finally, let us prove the convergence of the successive approximation $(\tilde{u}^{(j)(m)}, \tilde{q}^{(j)(m)})$ for $T \leqslant T_0$. Set

$$U^{(j)(m+1)} = \tilde{u}^{(j)(m+1)} - \tilde{u}^{(j)(m)}, \qquad Q^{(j)(m+1)} = \tilde{q}^{(j)(m+1)} - \tilde{q}^{(j)(m)},$$

and define

$$\begin{aligned} Y_m^*(T) \overset{\text{def}}{=} \sum_{j=1}^{2} \big(\|U^{(j)(m)}\|_{W_2^{l+2,l/2+1}(Q_T^{(j)})} + \|\nabla Q^{(j)(m)}\|_{W_2^{l,l/2}(Q_T^{(j)})} + \|Q^{(j)(m)}\|_{W_2^{l,l/2}(Q_T^{(j)})}\big) \\ + \|Q^{(1)(m)}\|_{W_2^{l+1/2,l/2+1/4}(S_{F,T}^{(1)})} + \|Q^{(1)(m)} - Q^{(2)(m)}\|_{W_2^{l+1/2,l/2+1/4}(S_{F,T}^{(2)})}, \end{aligned}$$

with $Y_1^*(T) = E_0^*$.

Following the proof of the uniform estimates, we can obtain

$$Y_{m+1}^*(T) \leqslant C\big(T, \delta, Y^*\big) Y_m^*(T),$$

where $C(T, \delta, Y^*)$ is a positive constant depending on T, δ, Y^* increasingly and $C(T, \delta, Y^*) \to 0$ as $T \to 0$. So we can take T^* small enough so that

$$C\big(T, \delta, Y^*\big) \leqslant 1/2 < 1, \quad \forall T \leqslant T^*.$$

Consequently,

$$\sum_{m=1}^{\infty} Y_m^*(T) < \infty, \quad \forall T \leqslant T^*,$$

which implies that

$$\big\{\tilde{u}^{(j)(m)}, \tilde{q}^{(j)(m)}\big\} \to \big\{\tilde{u}^{(j)}, \tilde{q}^{(j)}\big\}, \quad \forall t \leqslant T^*,$$

and $\{\tilde{u}^{(j)}, \tilde{q}^{(j)}\}$ satisfies the problem (4.2)–(4.3). Thus,

$$\big\{u^{(j)(m)}, q^{(j)(m)}\big\} \to \big\{u^{(j)}, q^{(j)}\big\}, \quad \forall t \leqslant T^*,$$

and $\{u^{(j)}, q^{(j)}\}$ satisfies the problem (1.8)–(1.9) and the inequality (1.10). The same argument gives the uniqueness of the solution to (1.8)–(1.9).

5. Global existence

This section is devoted to the proof of Theorem 1.2. We follow the proof of [17,20,22].

5.1. Auxiliary linearized problem

The auxiliary linearized problem of (1.8)–(1.9) is

$$\begin{cases} \rho_0^{(j)} u_t^{(j)} - \nu^{(j)} \nabla^2_{w^{(j)}} u^{(j)} + \nabla_{w^{(j)}} q^{(j)} = \rho_0^{(j)} f^{(j)} \quad \text{in } Q_T^{(j)}, \\ \nabla_{w^{(j)}} \cdot u^{(j)} = g^{(j)} \quad \text{in } Q_T^{(j)}, \\ u^{(j)}\big|_{t=0} = v_0^{(j)}(\xi), \quad \xi \in \Omega^{(j)},\ j = 1, 2, \end{cases} \tag{5.1}$$

together with the boundary conditions

$$\begin{cases} \Pi^{(1)}_{w^{(1)}} \big[\nu^{(1)} \mathrm{D}_{w^{(1)}} \big(u^{(1)}\big) n^{(1)}_{w^{(1)}}\big]\big|_{S^{(1)}_{F,T}} = \vec{b}^{(1)}, \\ -q^{(1)} + 2\nu^{(1)} n^{(1)}_{w^{(1)}} \cdot \big(\nabla_{w^{(1)}} u^{(1)} n^{(1)}_{w^{(1)}}\big) - \sigma^{(1)} \Delta^{(1)}_{w^{(1)}}(t) \displaystyle\int_0^t u^{(1)}\, d\tau \cdot n^{(1)}_{w^{(1)}} \Big|_{S^{(1)}_{F,T}} \\ \quad = b^{(1)} + \sigma^{(1)} \displaystyle\int_0^t B^{(1)}\, d\tau, \\ u^{(1)} - u^{(2)}\big|_{S^{(2)}_{F,T}} = 0, \\ \Pi^{(2)}_{w^{(2)}} \big[\nu^{(2)} \mathrm{D}_{w^{(2)}} \big(u^{(2)}\big) n^{(2)}_{w^{(2)}}\big] - \Pi^{(2)}_{w^{(1)}} \big[\nu^{(1)} \mathrm{D}_{w^{(1)}} \big(u^{(1)}\big) n^{(2)}_{w^{(1)}}\big]\big|_{S^{(2)}_{F,T}} = \vec{b}^{(2)}, \\ q^{(1)} - q^{(2)} + 2\nu^{(2)} n^{(2)}_{w^{(2)}} \cdot \big(\nabla_{w^{(2)}} u^{(2)} n^{(2)}_{w^{(2)}}\big) - 2\nu^{(1)} n^{(2)}_{w^{(1)}} \cdot \big(\nabla_{w^{(1)}} u^{(1)} n^{(2)}_{w^{(1)}}\big) \\ \quad - \dfrac{\sigma^{(2)}}{2} \displaystyle\sum_{j=1}^{2} \Delta^{(2)}_{w^{(j)}}(t) \int_0^t u^{(j)}\, d\tau \cdot n^{(2)}_{w^{(j)}} \Big|_{S^{(2)}_{F,T}} \\ \quad = b^{(2)} + \dfrac{\sigma^{(2)}}{2} \displaystyle\int_0^t B^{(2)}\, d\tau, \\ u^{(2)}\big|_{S_{B,T}} = 0, \end{cases} \tag{5.2}$$

where $w^{(j)}$ $(j = 1, 2)$ are given functions.

From the proof of Section 4.2, we can conclude that

Theorem 5.1. *Let $l, \sigma^{(j)}, \nu^{(j)}, \rho_0^{(j)}, v_0^{(j)}, S_B$ $(j = 1, 2)$ be as in Theorem 1.1 and $S_F^{(j)} \in W_2^{l+3/2}$. Let $f^{(j)} \in W_2^{l,l/2}(Q_T^{(j)})$, $g^{(j)} = \nabla \cdot R^{(j)} \in W_2^{l+1,l/2+1/2}(Q_T^{(j)})$ with $R^{(j)} \in W_2^{0,l/2+1}(Q_T^{(j)})$, $(\vec{b}^{(j)}, b^{(j)}) \in W_2^{l+1/2,l/2+1/4}(S^{(j)}_{F,T})$, $B^{(j)} \in W_2^{l-1/2,l/2-1/4}(S^{(j)}_{F,T})$ $(j = 1, 2)$, and the compatibility conditions analogues to Theorem 1.1 hold. We assume that*

$$T^{1/2-l/2} \big\| w^{(j)} \big\|_{W_2^{l+2,l/2+1}(Q_T^{(j)})} \leqslant \delta, \quad j = 1, 2, \tag{5.3}$$

with $\delta \ll 1$. Then the problem (5.1)–(5.2) *has a unique solution* $(u^{(1)}, q^{(1)}, u^{(2)}, q^{(2)})$ *such that* $u^{(j)} \in W_2^{l+2,l/2+1}(Q_T^{(j)})$, $q^{(j)}, \nabla q^{(j)} \in W_2^{l,l/2}(Q_T^{(j)})$, $q^{(1)}|_{S_F^{(1)}} \in W_2^{l+1/2,l/2+1/4}(S_{F,T}^{(1)})$, $q^{(1)} - q^{(2)}|_{S_F^{(2)}} \in W_2^{l+1/2,l/2+1/4}(S_{F,T}^{(2)})$ *and*

$$\begin{aligned}
&\sum_{j=1}^{2}\left(\|u^{(j)}\|_{W_2^{l+2,l/2+1}(Q_T^{(j)})} + \|q^{(j)}\|_{W_2^{l,l/2}(Q_T^{(j)})} + \|\nabla q^{(j)}\|_{W_2^{l,l/2}(Q_T^{(j)})}\right)\\
&\quad + \|q^{(1)}\|_{W_2^{l+1/2,l/2+1/4}(S_{F,T}^{(1)})} + \|q^{(1)} - q^{(2)}\|_{W_2^{l+1/2,l/2+1/4}(S_{F,T}^{(2)})}\\
&\leqslant C(T)\sum_{j=1}^{2}\Big[\|f^{(j)}\|_{W_2^{l,l/2}(Q_T^{(j)})} + \|g^{(j)}\|_{W_2^{l+1,l/2+1/2}(Q_T^{(j)})}\\
&\quad + \|R^{(j)}\|_{W_2^{0,l/2+1}(Q_T^{(j)})} + \big\|\big(\vec{b}^{(j)}, b^{(j)}\big)\big\|_{W_2^{l+1/2,l/2+1/4}(S_{F,T}^{(j)})}\\
&\quad + \sigma^{(j)}\|B^{(j)}\|_{W_2^{l-1/2,l/2-1/4}(S_{F,T}^{(j)})} + \|v_0^{(j)}\|_{W_2^{l+1}(\Omega^{(j)})}\Big].
\end{aligned} \tag{5.4}$$

5.2. A priori estimates

Let us begin with the energy equality of (1.1)–(1.2).

Lemma 5.1. *Let* $(\rho^{(j)}, v^{(j)})$ *satisfy* (1.1)–(1.2) *with* $f^{(j)} = 0$, *then there holds*

$$\begin{aligned}
&\frac{d}{dt}\sum_{j=1}^{2}\left(\frac{1}{2}\int_{\Omega^{(j)}(t)} \rho^{(j)}\left|v^{(j)}\right|^2 dx + \int_{\mathbb{R}^2}\left(\sqrt{1+\left|\nabla' F^{(j)}\right|^2} - 1\right) dx'\right.\\
&\quad \left. + \frac{g_0}{2}\int_{\mathbb{R}^2}\left|F^{(j)} - h^{(j)}\right|^2 dx'\right) + \frac{1}{2}\sum_{j=1}^{2}\nu^{(j)}\int_{\Omega^{(j)}(t)}\left|\mathrm{D}\big(v^{(j)}\big)\right|^2 dx = 0.
\end{aligned} \tag{5.5}$$

Proof. From Eqs. (1.1)–(1.2) with $f^{(j)} = 0$, we can obtain

$$\begin{aligned}
&\frac{d}{dt}\sum_{j=1}^{2}\frac{1}{2}\int_{\Omega^{(j)}(t)} \rho^{(j)}\left|v^{(j)}\right|^2 dx\\
&= -\frac{1}{2}\sum_{j=1}^{2}\nu^{(j)}\int_{\Omega^{(j)}(t)}\left|\mathrm{D}\big(v^{(j)}\big)\right|^2 dx + \int_{S_F^{(1)}(t)}\big(\sigma^{(1)}H^{(1)} - g_0\big(x_3 - h^{(1)}\big)\big)n^{(1)}\cdot v^{(1)}\, dS^{(1)}\\
&\quad + \int_{S_F^{(2)}(t)}\big(\sigma^{(2)}H^{(2)} - g_0\big(x_3 - h^{(2)}\big)\big)n^{(2)}\cdot v^{(2)}\, dS^{(2)},
\end{aligned}$$

where $dS^{(j)}$ is the surface element on $S_F^{(j)}(t)$.

Note that

$$n^{(j)} = {}^t\left(\frac{-\frac{\partial F^{(j)}}{\partial x_1}}{\sqrt{1+|\nabla' F^{(j)}|^2}}, \frac{-\frac{\partial F^{(j)}}{\partial x_2}}{\sqrt{1+|\nabla' F^{(j)}|^2}}, \frac{1}{\sqrt{1+|\nabla' F^{(j)}|^2}}\right)$$

and

$$\left(\frac{D}{Dt}\right)^{(j)}\big(x_3 - F^{(j)}\big) = 0,$$

we find that

$$v^{(j)} \cdot n^{(j)} = \frac{\frac{\partial F^{(j)}}{\partial t}}{\sqrt{1 + |\nabla' F^{(j)}|^2}},$$

on the other hand,

$$H^{(j)} = \nabla' \cdot \frac{\nabla' F^{(j)}}{\sqrt{1 + |\nabla' F^{(j)}|^2}}, \tag{5.6}$$

we get

$$\begin{aligned}\int\limits_{S_F^{(j)}(t)} H^{(j)} n^{(j)} \cdot v^{(j)}\, dS^{(j)} &= \int\limits_{\mathbb{R}^2} \left(\nabla' \cdot \frac{\nabla' F^{(j)}}{\sqrt{1 + |\nabla' F^{(j)}|^2}}\right) \frac{\partial F^{(j)}}{\partial t}\, dx' \\ &= -\frac{d}{dt} \int\limits_{\mathbb{R}^2} \left(\sqrt{1 + \left|\nabla' F^{(j)}\right|^2} - 1\right) dx',\end{aligned}$$

and

$$\begin{aligned}\int\limits_{S_F^{(j)}(t)} g_0\big(x_3 - h^{(j)}\big) n^{(j)} \cdot v^{(j)}\, dS^{(j)} &= \int\limits_{\mathbb{R}^2} g_0\big(F^{(j)} - h^{(j)}\big) \frac{\partial F^{(j)}}{\partial t}\, dx' \\ &= \frac{g_0}{2} \frac{d}{dt} \int\limits_{\mathbb{R}^2} \left|F^{(j)} - h^{(j)}\right|^2 dx'.\end{aligned}$$

This completes the lemma. □

Next, we consider the equation for $F^{(j)}$:

$$\sigma^{(j)} \nabla' \cdot \frac{\nabla' F^{(j)}}{\sqrt{1 + |\nabla' F^{(j)}|^2}} - g_0\big(F^{(j)} - h^{(j)}\big) = \Phi^{(j)} \quad \text{on } \mathbb{R}^2. \tag{5.7}$$

We omit the index $^{(j)}$ in the following lemma.

Lemma 5.2. *(See [17,20].) Let* $F(\cdot,t) - h \in W_2^{l+5/2}(\mathbb{R}^2)$ $(1/2 < l < 1)$ *be a solution to* (5.7) *satisfying the condition*

$$\left\| F(\cdot,t) - h \right\|_{W_2^{l+3/2}(\mathbb{R}^2)} \leqslant \delta, \tag{5.8}$$

with $\delta \ll 1$, *then there hold*:

(i) *if* $\Phi \in W_2^{l-1/2}(\mathbb{R}^2)$, *then*

$$\|F(\cdot,t)-h\|_{W_2^{l+3/2}(\mathbb{R}^2)} \leqslant c_1\|\Phi\|_{W_2^{l-1/2}(\mathbb{R}^2)} + c_2\|F(\cdot,t)-h\|_{L^2(\mathbb{R}^2)};$$

(ii) *if* $\Phi \in W_2^{l+1/2}(\mathbb{R}^2)$, *then*

$$\|F(\cdot,t)-h\|_{W_2^{l+5/2}(\mathbb{R}^2)} \leqslant c_3\|\Phi\|_{W_2^{l+1/2}(\mathbb{R}^2)} + c_4\|F(\cdot,t)-h\|_{L^2(\mathbb{R}^2)},$$

where the constants c_2, c_4 *may depend on* $\|F(\cdot,t)-h\|_{W_2^{l+3/2}(\mathbb{R}^2)}$.

In what follows, we assume $T=1$ without loss of generality.

Lemma 5.3. *Let* $u^{(j)} \in W_2^{l+2,l/2+1}(Q_1^{(j)})$, $q^{(j)}, \nabla q^{(j)} \in W_2^{l,l/2}(Q_1^{(j)})$ $(j=1,2)$, $q^{(1)}|_{S_F^{(1)}} \in W_2^{l+1/2,l/2+1/4}(S_{F,1}^{(1)})$, $q^{(1)}-q^{(2)}|_{S_F^{(2)}} \in W_2^{l+1/2,l/2+1/4}(S_{F,1}^{(2)})$ *be the solution to the problem* (1.8)–(1.9) *with* $f^{(j)}=0$ *satisfying condition* (5.3) *with* $\delta \ll 1$. *Suppose that* $F^{(j)}(\cdot,t)-h^{(j)} \in W_2^{l+3/2}(\mathbb{R}^2)$ *satisfies* (5.8) *with* $\delta \ll 1$. *Then we have*

$$\begin{aligned} U(\lambda) &\overset{\text{def}}{=} \sum_{j=1}^{2}\big(\|u^{(j)}\|_{W_2^{l+2,l/2+1}(Q^{(j)}(\lambda))} + \|q^{(j)}\|_{W_2^{l,l/2}(Q^{(j)}(\lambda))} + \|\nabla q^{(j)}\|_{W_2^{l,l/2}(Q^{(j)}(\lambda))}\big) \\ &\quad + \|q^{(1)}\|_{W_2^{l+1/2,l/2+1/4}(S_F^{(1)}(\lambda))} + \|q^{(1)}-q^{(2)}\|_{W_2^{l+1/2,l/2+1/4}(S_F^{(2)}(\lambda))} \\ &\leqslant C\lambda^{-\frac{4(l+2)}{l-1/2}} \sum_{j=1}^{2}\big(\|u^{(j)}\|_{L^2(Q^{(j)}(0))} + \sigma^{(j)}\|F^{(j)}-h^{(j)}\|_{L^2(\mathbb{R}^2\times(t_0,1))}\big), \end{aligned} \tag{5.9}$$

where $0<\lambda<1$, $t_0>0$, $t_0+\lambda<1$,

$$Q^{(j)}(\lambda) = \Omega^{(j)} \times (t_0+\lambda, 1), \qquad S_F^{(j)}(\lambda) = S_F^{(j)} \times (t_0+\lambda, 1),$$

and for any $t_1 > t_0$,

$$\begin{aligned} &\sup_{t_1<t<1} \sum_{j=1}^{2}\big(\|u^{(j)}\|_{W_2^{l+2}(\Omega^{(j)})} + \|q^{(j)}\|_{W_2^{l+1}(\Omega^{(j)})}\big) \\ &\quad \leqslant c(t_1)\sum_{j=1}^{2}\big(\|u^{(j)}\|_{L^2(Q^{(j)}(0))} + \sigma^{(j)}\|F^{(j)}-h^{(j)}\|_{L^2(\mathbb{R}^2\times(t_0,1))}\big), \end{aligned} \tag{5.10}$$

where $c(t_1)$ *is a constant depending on* $(t_1-t_0)^{-1}$.

Proof. We introduce smooth cut-off functions $\zeta_\lambda(t) \in [0,1]$ such that

$$\zeta_\lambda(t) = \begin{cases} 1, & t \geqslant t_0+\lambda, \\ 0, & t \leqslant t_0+\lambda/2, \end{cases}$$

$$\left|\left(\frac{d}{dt}\right)^k \zeta_\lambda(t)\right| \leqslant C_k\lambda^{-k}$$

and for the given $(u^{(j)}, q^{(j)})$, we set $u_\lambda^{(j)} = u^{(j)}\zeta_\lambda$, $q_\lambda^{(j)} = q^{(j)}\zeta_\lambda$. Then $(u_\lambda^{(j)}, q_\lambda^{(j)})$ satisfies the following linear problem:

$$\begin{cases} \rho_0^{(j)} \dfrac{\partial}{\partial t} u_\lambda^{(j)} - \nu^{(j)} \nabla_{u^{(j)}}^2 u_\lambda^{(j)} + \nabla_{u^{(j)}} q_\lambda^{(j)} = \rho_0^{(j)} u^{(j)} \dfrac{d\zeta_\lambda}{dt} \quad \text{in } Q_1^{(j)}, \\ \nabla_{u^{(j)}} \cdot u_\lambda^{(j)} = 0 \quad \text{in } Q_1^{(j)}, \\ u_\lambda^{(j)}\big|_{t=0} = 0 \quad \text{in } \Omega^{(j)},\ j = 1, 2, \end{cases} \tag{5.11}$$

together with the boundary conditions

$$\begin{cases} \Pi_{u^{(1)}}^{(1)}\big[\nu^{(1)} \mathrm{D}_{u^{(1)}}\big(u_\lambda^{(1)}\big) n_{u^{(1)}}^{(1)}\big]\big|_{S_{F,1}^{(1)}} = 0, \\ -q_\lambda^{(1)} + 2\nu^{(1)} n_{u^{(1)}}^{(1)} \cdot \big(\nabla_{u^{(1)}} u_\lambda^{(1)} n_{u^{(1)}}^{(1)}\big) - \sigma^{(1)} \displaystyle\int_0^t n_{u^{(1)}}^{(1)} \cdot \Delta_{u^{(1)}}^{(1)}(\tau) u_\lambda^{(1)}\, d\tau \bigg|_{S_{F,1}^{(1)}} \\ \quad = \displaystyle\int_0^t D^{(1)}\, d\tau, \\ u_\lambda^{(1)} - u_\lambda^{(2)}\big|_{S_{F,1}^{(2)}} = 0, \\ \Pi_{u^{(2)}}^{(2)}\big[\nu^{(2)} \mathrm{D}_{u^{(2)}}\big(u_\lambda^{(2)}\big) n_{u^{(2)}}^{(2)}\big] - \Pi_{u^{(1)}}^{(2)}\big[\nu^{(1)} \mathrm{D}_{u^{(1)}}\big(u_\lambda^{(1)}\big) n_{u^{(1)}}^{(2)}\big]\big|_{S_{F,1}^{(2)}} = 0, \\ q_\lambda^{(1)} - q_\lambda^{(2)} + 2\nu^{(2)} n_{u^{(2)}}^{(2)} \cdot \big(\nabla_{u^{(2)}} u_\lambda^{(2)} n_{u^{(2)}}^{(2)}\big) - 2\nu^{(1)} n_{u^{(1)}}^{(2)} \cdot \big(\nabla_{u^{(1)}} u_\lambda^{(1)} n_{u^{(1)}}^{(2)}\big) \\ \quad - \dfrac{\sigma^{(2)}}{2} \displaystyle\sum_{j=1}^2 \int_0^t n_{u^{(j)}}^{(2)} \cdot \Delta_{u^{(j)}}^{(2)}(\tau) u_\lambda^{(j)}\, d\tau \bigg|_{S_{F,1}^{(2)}} = \int_0^t D^{(2)}\, d\tau, \\ u_\lambda^{(2)}\big|_{S_{B,1}} = 0, \end{cases} \tag{5.12}$$

with

$$\begin{aligned} D^{(1)} &= \frac{d\zeta_\lambda}{dt}\big[-q^{(1)} + 2\nu^{(1)} n_{u^{(1)}}^{(1)} \cdot \big(\nabla_{u^{(1)}} u^{(1)} n_{u^{(1)}}^{(1)}\big)\big] \\ &\quad + \sigma^{(1)} \zeta_\lambda \left[\frac{\partial}{\partial t} n_{u^{(1)}}^{(1)} \cdot \Delta_{u^{(1)}}^{(1)} X_{u^{(1)}} + n_{u^{(1)}}^{(1)} \cdot \dot{\Delta}_{u^{(1)}}^{(1)} X_{u^{(1)}}\right] - g_0 \zeta_\lambda u_3^{(1)}, \\ D^{(2)} &= \frac{d\zeta_\lambda}{dt}\big[q^{(1)} - q^{(2)} + 2\nu^{(2)} n_{u^{(2)}}^{(2)} \cdot \big(\nabla_{u^{(2)}} u^{(2)} n_{u^{(2)}}^{(2)}\big) - 2\nu^{(1)} n_{u^{(1)}}^{(2)} \cdot \big(\nabla_{u^{(1)}} u^{(1)} n_{u^{(1)}}^{(2)}\big)\big] \\ &\quad + \frac{\sigma^{(2)}}{2} \sum_{j=1}^2 \zeta_\lambda \left[\frac{\partial}{\partial t} n_{u^{(j)}}^{(2)} \cdot \Delta_{u^{(j)}}^{(2)} X_{u^{(j)}} + n_{u^{(j)}}^{(2)} \cdot \dot{\Delta}_{u^{(j)}}^{(2)} X_{u^{(j)}}\right] - g_0 \zeta_\lambda u_3^{(2)}. \end{aligned}$$

Here we have used the formula

$$\begin{aligned} &-q_\lambda^{(1)} + 2\nu^{(1)} n_{u^{(1)}}^{(1)} \cdot \big(\nabla_{u^{(1)}} u_\lambda^{(1)} n_{u^{(1)}}^{(1)}\big) \\ &\quad = \int_0^t \frac{d}{d\tau} \big\{\zeta_\lambda\big[-q^{(1)} + 2\nu^{(1)} n_{u^{(1)}}^{(1)} \cdot \big(\nabla_{u^{(1)}} u^{(1)} n_{u^{(1)}}^{(1)}\big)\big]\big\}\, d\tau \end{aligned}$$

$$= \int_0^t \frac{d}{d\tau}\zeta_\lambda\big[-q^{(1)} + 2\nu^{(1)} n^{(1)}_{u^{(1)}} \cdot \big(\nabla_{u^{(1)}} u^{(1)} n^{(1)}_{u^{(1)}}\big)\big]\,d\tau + \sigma^{(1)} \int_0^t \zeta_\lambda\left[\frac{\partial}{\partial t} n^{(1)}_{u^{(1)}} \cdot \Delta^{(1)}_{u^{(1)}} X_{u^{(1)}}\right.$$

$$\left. + n^{(1)}_{u^{(1)}} \cdot \dot{\Delta}^{(1)}_{u^{(1)}} X_{u^{(1)}}\right] d\tau + \sigma^{(1)} \int_0^t n^{(1)}_{u^{(1)}} \cdot \Delta^{(1)}_{u^{(1)}}(\tau) u^{(1)}_\lambda \, d\tau - \int_0^t g_0 \zeta_\lambda u^{(1)}_3 \, d\tau$$

together with a similar formula for

$$q^{(1)}_\lambda - q^{(2)}_\lambda + 2\nu^{(2)} n^{(2)}_{u^{(2)}} \cdot \big(\nabla_{u^{(2)}} u^{(2)}_\lambda n^{(2)}_{u^{(2)}}\big) - 2\nu^{(1)} n^{(2)}_{u^{(1)}} \cdot \big(\nabla_{u^{(1)}} u^{(1)}_\lambda n^{(2)}_{u^{(1)}}\big).$$

Although the form of the boundary conditions (5.12) is slightly different from the boundary conditions of the auxiliary linear problem (5.1)–(5.2), it is easy to find from the construction of the solution of (3.1)–(3.2), that the same results such as Theorem 5.1 hold for the problem (5.11)–(5.12). Thus,

$$\begin{aligned} &\sum_{j=1}^2 \big(\|u^{(j)}_\lambda\|_{W_2^{l+2,l/2+1}(Q_1^{(j)})} + \|q^{(j)}_\lambda\|_{W_2^{l,l/2}(Q_1^{(j)})} + \|\nabla q^{(j)}_\lambda\|_{W_2^{l,l/2}(Q_1^{(j)})}\big) \\ &\quad + \|q^{(1)}_\lambda\|_{W_2^{l+1/2,l/2+1/4}(S^{(1)}_{F,1})} + \|q^{(1)}_\lambda - q^{(2)}_\lambda\|_{W_2^{l+1/2,l/2+1/4}(S^{(2)}_{F,1})} \\ &\quad \leqslant C \sum_{j=1}^2 \left[\left\|u^{(j)} \frac{d\zeta_\lambda}{dt}\right\|_{W_2^{l,l/2}(Q_1^{(j)})} + \|D^{(j)}\|_{W_2^{l-1/2,l/2-1/4}(S^{(j)}_{F,1})}\right]. \end{aligned} \tag{5.13}$$

We shall estimate each term on the right-hand side of (5.13) and we omit the index ${}^{(j)}$ for a while. Firstly, we can easily get

$$\begin{aligned} \left\|u \frac{d\zeta_\lambda}{dt}\right\|^2_{W_2^{l,0}(Q_1)} &= \int_{t_0+\frac{\lambda}{2}}^1 \|u\|^2_{W_2^l(\Omega)} \left|\frac{d\zeta_\lambda}{dt}\right|^2 dt \leqslant C\lambda^{-2}\|u\|^2_{W_2^{l,0}(Q(\frac{\lambda}{2}))}, \\ \left\|u \frac{d\zeta_\lambda}{dt}\right\|^2_{W_2^{0,l/2}(Q_1)} &\leqslant C \int_\Omega \left(\|u(x,\cdot)\|_{W_2^{l/2}(t_0+\frac{\lambda}{2},1)} \left\|\frac{d\zeta_\lambda}{dt}\right\|_{L^\infty(0,1)} \right. \\ &\quad \left. + \|u(x,\cdot)\|_{L^2(t_0+\frac{\lambda}{2},1)} \left\|\frac{d\zeta_\lambda}{dt}\right\|_{\dot W_\infty^{l/2+\bar\delta}(t_0+\frac{\lambda}{2},1)}\right)^2 dx \\ &\leqslant C\lambda^{-2}\|u\|^2_{W_2^{0,l/2}(Q(\frac{\lambda}{2}))} + C\lambda^{-2(1+l/2+\bar\delta)}\|u\|^2_{L^2(Q(\frac{\lambda}{2}))}, \end{aligned}$$

which imply

$$\begin{aligned} \left\|u \frac{d\zeta_\lambda}{dt}\right\|_{W_2^{l,l/2}(Q_1)} &\leqslant C\lambda^{-1}\|u\|_{W_2^{l,l/2}(Q(\frac{\lambda}{2}))} + C\lambda^{-(1+l/2+\bar\delta)}\|u\|_{L^2(Q(\frac{\lambda}{2}))} \\ &\leqslant \epsilon_1\|u\|_{W_2^{l+2,l/2+1}(Q(\frac{\lambda}{2}))} + C\lambda^{-(1+l/2+\bar\delta)}\epsilon_1^{-l/2}\|u\|_{L^2(Q(\frac{\lambda}{2}))}, \end{aligned} \tag{5.14}$$

where $\bar\delta \in (0, l/2 - 1/4)$, $\epsilon_1 \in (0,1)$, and we have used the interpolation inequality in the last inequality.

Now, we estimate $D^{(1)}$ and $D^{(2)}$. Similar to (5.14), we can obtain

$$\left\|\frac{d\zeta_\lambda}{dt}q\right\|_{W_2^{l-1/2,l/2-1/4}(S_{F,1})} \leqslant \epsilon_1\|q\|_{W_2^{l+1/2,l/2+1/4}(S_F(\frac{\lambda}{2}))} + C\epsilon_1^{-(l-1/2)}\lambda^{-(l+1/2)}\|q\|_{L^2(S_F(\frac{\lambda}{2}))},$$

$$\|\zeta_\lambda u_3\|_{W_2^{l-1/2,l/2-1/4}(S_{F,1})} \leqslant \epsilon_1\|u\|_{W_2^{l+2,l/2+1}(Q(\frac{\lambda}{2}))} + C\lambda^{-(l/2-1/4+\bar{\delta})(l+2)}\epsilon_1^{-l/2}\|u\|_{L^2(Q(\frac{\lambda}{2}))},$$

and

$$\begin{aligned}
&\left\|\frac{d\zeta_\lambda}{dt}n_u\cdot(\nabla_u u n_u)\right\|_{W_2^{l-1/2,l/2-1/4}(S_{F,1})} \\
&\quad\leqslant C\lambda^{-1}\left\|n_u\cdot(\nabla_u u n_u)\right\|_{W_2^{l-1/2,l/2-1/4}(S_F(\frac{\lambda}{2}))} + C\lambda^{-1-(l/2-1/4+\bar{\delta})}\left\|n_u\cdot(\nabla_u u n_u)\right\|_{L^2(S_F(\frac{\lambda}{2}))} \\
&\quad\leqslant C\lambda^{-l/2-3/4-\bar{\delta}}\|\nabla u\|_{W_2^{l,l/2}(Q(\frac{\lambda}{2}))} \\
&\quad\leqslant \epsilon_1\|u\|_{W_2^{l+2,l/2+1}(Q(\frac{\lambda}{2}))} + C\lambda^{-(l/2+3/4+\bar{\delta})(l+2)}\epsilon_1^{-(l+1)}\|u\|_{L^2(Q(\frac{\lambda}{2}))},
\end{aligned}$$

where we used Lemma 4.2 and the interpolation inequality.

Thanks to Lemmas 4.2–4.3, we can calculate as that in Step 5 of Section 4.2 and get

$$\begin{aligned}
&\left\|\zeta_\lambda\frac{\partial}{\partial t}n_u\cdot\Delta_u\xi\right\|_{W_2^{l-1/2,l/2-1/4}(S_{F,1})} + \|\zeta_\lambda n_u\cdot\dot{\Delta}_u\xi\|_{W_2^{l-1/2,l/2-1/4}(S_{F,1})} \\
&\quad\leqslant C\lambda^{-l/2+1/4-\bar{\delta}}\big(\|u\|_{W_2^{l+2-2\theta,l/2+1-\theta}(Q(\frac{\lambda}{2}))} + \|\nabla u\|_{W_2^{l,l/2}(Q(\frac{\lambda}{2}))}\big) \\
&\quad\leqslant \epsilon_1\|u\|_{W_2^{l+2,l/2+1}(Q(\frac{\lambda}{2}))} + c(\epsilon_1)\lambda^{-(l+2)}\|u\|_{L^2(Q(\frac{\lambda}{2}))},
\end{aligned}$$

where $\theta\in(0,l/2-1/4)$ and we used interpolation inequality in the last inequality. Meanwhile, thanks to Lemma 4.2, (4.9)–(4.11) and (5.3), we obtain that

$$\begin{aligned}
&\left\|\zeta_\lambda\frac{\partial}{\partial t}n_u\cdot\Delta_u\int_0^t u\,d\tau\right\|_{W_2^{l-1/2,l/2-1/4}(S_{F,1})} \\
&\quad\leqslant C\delta\|u_\lambda\|_{W_2^{l+2,l/2+1}(Q_1)} + \epsilon_1\|u\|_{W_2^{l+2,l/2+1}(Q(\frac{\lambda}{2}))} + c(\epsilon_1)\lambda^{-(l+1)}\|u\|_{L^2(Q(\frac{\lambda}{2}))},
\end{aligned}$$

and $\|\zeta_\lambda n_u\cdot\dot{\Delta}_u\int_0^t u\,d\tau\|_{W_2^{l-1/2,l/2-1/4}(S_{F,1})}$ has the same estimate as above. So we can obtain the desired bound of $\sum_{j=1}^2\|D^{(j)}\|_{W_2^{l-1/2,l/2-1/4}(S_{F,1}^{(j)})}$. Therefore, we get by (5.13) that

$$\begin{aligned}
U(\lambda) \leqslant{}& \epsilon_2 U(\lambda/2) + c(\epsilon_2)\lambda^{-(l/2+3/4+\bar{\delta})(l+2)}\Bigg(\sum_{j=1}^2\left\|u^{(j)}\right\|_{L^2(Q^{(j)}(\frac{\lambda}{2}))} + \left\|q^{(1)}\right\|_{L^2(S_F^{(1)}(\frac{\lambda}{2}))} \\
&+ \left\|q^{(1)}-q^{(2)}\right\|_{L^2(S_F^{(2)}(\frac{\lambda}{2}))}\Bigg).
\end{aligned}\tag{5.15}$$

Next, it remains to estimate

$$\|q^{(1)}\|_{L^2(S_F^{(1)}(\frac{\lambda}{2}))}+\|q^{(1)}-q^{(2)}\|_{L^2(S_F^{(2)}(\frac{\lambda}{2}))}.$$

From the boundary conditions $(1.2)_1$ and $(1.2)_3$, we get by Lemma 4.2 that

$$\begin{aligned}
&\|q^{(1)}\|_{L^2(S_F^{(1)}(\frac{\lambda}{2}))}+\|q^{(1)}-q^{(2)}\|_{L^2(S_F^{(2)}(\frac{\lambda}{2}))}\\
&\quad\leqslant C\sum_{j=1}^{2}\big(\|\nabla u^{(j)}\|_{L^2(S_F^{(j)}(\frac{\lambda}{2}))}\\
&\qquad+\sigma^{(j)}\|H^{(j)}\|_{L^2(t_0+\lambda/2,1;L^2(\mathbb{R}^2))}+g_0\|F^{(j)}-h^{(j)}\|_{L^2(t_0+\lambda/2,1;L^2(\mathbb{R}^2))}\big).
\end{aligned}\tag{5.16}$$

Thanks to Lemma 4.2, Lemma 5.2(i) and (5.6), we get by interpolation inequality that

$$\begin{aligned}
&\sum_{j=1}^{2}\|H^{(j)}\|_{L^2(t_0+\lambda/2,1;L^2(\mathbb{R}^2))}\\
&\quad\leqslant C\sum_{j=1}^{2}\|\nabla' F^{(j)}\|_{L^2(t_0+\lambda/2,1;H^1(\mathbb{R}^2))}\\
&\quad\leqslant \sum_{j=1}^{2}\big(\epsilon\|F^{(j)}-h^{(j)}\|_{L^2(t_0+\lambda/2,1;W_2^{l+3/2}(\mathbb{R}^2))}+C\epsilon^{-\frac{2}{l-1/2}}\|F^{(j)}-h^{(j)}\|_{L^2(t_0+\lambda/2,1;L^2(\mathbb{R}^2))}\big)\\
&\quad\leqslant \epsilon_1\sum_{j=1}^{2}\big(\|q^{(j)}\|_{W_2^{l+1/2,0}(S_F^{(j)}(\frac{\lambda}{2}))}+\|u^{(j)}\|_{W_2^{l+2,0}(Q^{(j)}(\frac{\lambda}{2}))}\big)\\
&\qquad+C\epsilon_1^{-\frac{2}{l-1/2}}\sum_{j=1}^{2}\|F^{(j)}-h^{(j)}\|_{L^2(t_0+\lambda/2,1;L^2(\mathbb{R}^2))},
\end{aligned}$$

which together with (5.16) and the interpolation inequality implies that

$$\begin{aligned}
&\|q^{(1)}\|_{L^2(S_F^{(1)}(\frac{\lambda}{2}))}+\|q^{(1)}-q^{(2)}\|_{L^2(S_F^{(2)}(\frac{\lambda}{2}))}\\
&\quad\leqslant \epsilon_3 U\left(\frac{\lambda}{2}\right)+C\epsilon_3^{-\frac{2}{l-1/2}}\sum_{j=1}^{2}\big(\|u^{(j)}\|_{L^2(Q^{(j)}(\frac{\lambda}{2}))}+\sigma^{(j)}\|F^{(j)}-h^{(j)}\|_{L^2(t_0+\lambda/2,1;L^2(\mathbb{R}^2))}\big).
\end{aligned}$$

Therefore, we get by (5.15) that

$$U(\lambda)\leqslant \epsilon_4 U\left(\frac{\lambda}{2}\right)+c(\epsilon_4)\lambda^{-\frac{4(l+2)}{l-1/2}}\sum_{j=1}^{2}\big(\|u^{(j)}\|_{L^2(Q^{(j)}(0))}+\sigma^{(j)}\|F^{(j)}-h^{(j)}\|_{L^2(t_0,1;L^2(\mathbb{R}^2))}\big).\tag{5.17}$$

If we take $\epsilon_4<2^{-\frac{4(l+2)}{l-1/2}}$ and use (5.17) again and again, then we can obtain

$$U(\lambda) \leqslant C\lambda^{-\frac{4(l+2)}{l-1/2}} \sum_{j=1}^{2} \left(\left\| u^{(j)} \right\|_{L^2(Q^{(j)}(0))} + \sigma^{(j)} \left\| F^{(j)} - h^{(j)} \right\|_{L^2(\mathbb{R}^2 \times (t_0,1))} \right), \tag{5.18}$$

that is the inequality (5.9).

In order to obtain (5.10), we consider the differences

$$u^{(j)(s)}(\xi,t) \stackrel{\mathrm{def}}{=} u_\lambda^{(j)}(\xi,t) - u_\lambda^{(j)}(\xi,t-s) \stackrel{\mathrm{def}}{=} u_\lambda^{(j)} - u'^{(j)}_\lambda,$$
$$q^{(j)(s)}(\xi,t) \stackrel{\mathrm{def}}{=} q_\lambda^{(j)}(\xi,t) - q_\lambda^{(j)}(\xi,t-s) \stackrel{\mathrm{def}}{=} q_\lambda^{(j)} - q'^{(j)}_\lambda,$$

where $s < t_0$. Subtracting from (5.11)–(5.12) the similar equations hold for $(u'^{(j)}_\lambda, q'^{(j)}_\lambda)$, and we obtain the system of equations for $(u^{(j)(s)}, q^{(j)(s)})$

$$\begin{cases} \rho_0^{(j)} \dfrac{\partial}{\partial t} u^{(j)(s)} - \nu^{(j)} \nabla^2_{u^{(j)}} u^{(j)(s)} + \nabla_{u^{(j)}} q^{(j)(s)} = \bar{f}^{(j)} \quad \text{in } Q_1^{(j)}, \\ \nabla_{u^{(j)}} \cdot u^{(j)(s)} = \bar{g}^{(j)} \quad \text{in } Q_1^{(j)}, \\ u^{(j)(s)}\big|_{t=0} = 0 \quad \text{in } \Omega^{(j)},\ j=1,2, \end{cases} \tag{5.19}$$

together with the boundary conditions

$$\begin{cases} \Pi_{u^{(1)}}^{(1)} \left[\nu^{(1)} \mathrm{D}_{u^{(1)}} \left(u^{(1)(s)} \right) n_{u^{(1)}}^{(1)} \right] \big|_{S_{F,1}^{(1)}} = \vec{d}^{(1)}, \\ -q^{(1)(s)} + 2\nu^{(1)} n_{u^{(1)}}^{(1)} \cdot \left(\nabla_{u^{(1)}} u^{(1)(s)} n_{u^{(1)}}^{(1)} \right) - \sigma^{(1)} \displaystyle\int_0^t n_{u^{(1)}}^{(1)} \cdot \Delta_{u^{(1)}}^{(1)}(\tau) u^{(1)(s)}\, d\tau \Big|_{S_{F,1}^{(1)}} \\ \quad = d^{(1)} + \displaystyle\int_0^t B^{(1)}\, d\tau, \\ u^{(1)(s)} - u^{(2)(s)} \big|_{S_{F,1}^{(2)}} = 0, \quad u^{(2)(s)} \big|_{S_{B,1}} = 0, \\ \Pi_{u^{(2)}}^{(2)} \left[\nu^{(2)} \mathrm{D}_{u^{(2)}} \left(u^{(2)(s)} \right) n_{u^{(2)}}^{(2)} \right] - \Pi_{u^{(1)}}^{(2)} \left[\nu^{(1)} \mathrm{D}_{u^{(1)}} \left(u^{(1)(s)} \right) n_{u^{(1)}}^{(2)} \right] \big|_{S_{F,1}^{(2)}} = \vec{d}^{(2)}, \\ q^{(1)(s)} - q^{(2)(s)} + 2\nu^{(2)} n_{u^{(2)}}^{(2)} \cdot \left(\nabla_{u^{(2)}} u^{(2)(s)} n_{u^{(2)}}^{(2)} \right) - 2\nu^{(1)} n_{u^{(1)}}^{(2)} \cdot \left(\nabla_{u^{(1)}} u^{(1)(s)} n_{u^{(1)}}^{(2)} \right) \\ \quad - \dfrac{\sigma^{(2)}}{2} \displaystyle\sum_{j=1}^{2} \int_0^t n_{u^{(j)}}^{(2)} \cdot \Delta_{u^{(j)}}^{(2)}(\tau) u^{(j)(s)}\, d\tau \Big|_{S_{F,1}^{(2)}} = d^{(2)} + \int_0^t B^{(2)}\, d\tau, \end{cases} \tag{5.20}$$

with

$$\bar{f}^{(j)} = \rho_0^{(j)} \left(u^{(j)} \frac{d\zeta_\lambda}{dt} - u'^{(j)} \frac{d\zeta'_\lambda}{dt} \right) + \nu^{(j)} \left(\nabla^2_{u^{(j)}} - \nabla^2_{u'^{(j)}} \right) u'^{(j)}_\lambda - \left(\nabla_{u^{(j)}} - \nabla_{u'^{(j)}} \right) q'^{(j)}_\lambda,$$
$$\bar{g}^{(j)} = \left(\nabla_{u'^{(j)}} - \nabla_{u^{(j)}} \right) u'^{(j)}_\lambda = \nabla \cdot \left[\left({}^t\mathcal{A}_{u'^{(j)}} - {}^t\mathcal{A}_{u^{(j)}} \right) \right],$$
$$\vec{d}^{(1)} = \Pi_{u^{(1)}}^{(1)} \left[-\nu^{(1)} \mathrm{D}_{u^{(1)}} \left(u'^{(1)}_\lambda \right) n_{u^{(1)}}^{(1)} + \Pi_{u'^{(1)}}^{(1)} \left[\nu^{(1)} \mathrm{D}_{u'^{(1)}} \left(u'^{(1)}_\lambda \right) n_{u'^{(1)}}^{(1)} \right] \right] \big|_{S_{F,1}^{(1)}},$$
$$d^{(1)} = 2\nu^{(1)} \left[n_{u'^{(1)}}^{(1)} \cdot \left(\nabla_{u'^{(1)}} u'^{(1)}_\lambda n_{u'^{(1)}}^{(1)} \right) - n_{u^{(1)}}^{(1)} \cdot \left(\nabla_{u^{(1)}} u'^{(1)}_\lambda n_{u^{(1)}}^{(1)} \right) \right] \big|_{S_{F,1}^{(1)}},$$

$$\begin{aligned}
B^{(1)} &= -\sigma^{(1)}\big[n^{(1)}_{u'^{(1)}}\cdot\Delta^{(1)}_{u'^{(1)}}(t-s) - n^{(1)}_{u^{(1)}}\cdot\Delta^{(1)}_{u^{(1)}}(t)\big]u'^{(1)}_{\lambda} \\
&\quad + \big[D^{(1)}(\xi,t) - D^{(1)}(\xi,t-s)\big]\big|_{S^{(1)}_{F,1}}, \\
\vec{d}^{(2)} &= \sum_{j=1}^{2}(-1)^j\nu^{(j)}\Pi^{(2)}_{u^{(j)}}\big[-\mathrm{D}_{u^{(j)}}\big(u'^{(j)}_{\lambda}\big)n^{(2)}_{u^{(j)}} + \Pi^{(2)}_{u'^{(j)}}\big[\mathrm{D}_{u'^{(j)}}\big(u'^{(j)}_{\lambda}\big)n^{(2)}_{u'^{(j)}}\big]\big]\big|_{S^{(2)}_{F,1}}, \\
d^{(2)} &= 2\sum_{j=1}^{2}(-1)^j\nu^{(j)}\big[n^{(2)}_{u'^{(j)}}\cdot\big(\nabla_{u'^{(j)}}u'^{(j)}_{\lambda}n^{(2)}_{u'^{(j)}}\big) - n^{(2)}_{u^{(j)}}\cdot\big(\nabla_{u^{(j)}}u'^{(j)}_{\lambda}n^{(2)}_{u^{(j)}}\big)\big]\big|_{S^{(2)}_{F,1}}, \\
B^{(2)} &= -\frac{\sigma^{(2)}}{2}\sum_{j=1}^{2}\big[n^{(2)}_{u'^{(j)}}\cdot\Delta^{(2)}_{u'^{(j)}}(t-s)u'^{(j)}_{\lambda} - n^{(2)}_{u^{(j)}}\cdot\Delta^{(2)}_{u^{(j)}}(t)u'^{(j)}_{\lambda}\big] \\
&\quad + \big[D^{(2)}(\xi,t) - D^{(2)}(\xi,t-s)\big]\big|_{S^{(2)}_{F,1}},
\end{aligned} \tag{5.21}$$

where $\zeta'_\lambda(t)=\zeta_\lambda(t-s)$, $u'^{(j)}(\xi,t)=u'^{(j)}(\xi,t-s)$.

Applying Theorem 5.1 to problem (5.19)–(5.20), we can obtain

$$\begin{aligned}
&\sum_{j=1}^{2}\big(\big\|u^{(j)(s)}\big\|_{W_2^{l+2,l/2+1}(Q^{(j)}(\lambda/2))} + \big\|\nabla q^{(j)(s)}\big\|_{W_2^{l,l/2}(Q^{(j)}(\lambda/2))}\big) \\
&\qquad + \big\|q^{(1)(s)}\big\|_{W_2^{l+1/2,l/2+1/4}(S_F^{(1)}(\lambda/2))} + \big\|q^{(1)(s)} - q^{(2)(s)}\big\|_{W_2^{l+1/2,l/2+1/4}(S_F^{(2)}(\lambda/2))} \\
&\quad \leqslant Cs^{\beta}\lambda^{-\frac{4(l+2)}{l-1/2}}\sum_{j=1}^{2}\big(\big\|u^{(j)}\big\|_{L^2(Q^{(j)}(0))} + \sigma^{(j)}\big\|F^{(j)} - h^{(j)}\big\|_{L^2(\mathbb{R}^2\times(t_0,T))}\big),
\end{aligned} \tag{5.22}$$

where $\beta>1/2$. In order to obtain (5.22), we need to estimate the quantities on (5.21). Here we only take the following term in $\bar{f}^{(j)}$ for example, and the other terms are similar. We omit the index $^{(j)}$ for a while.

$$\begin{aligned}
&\big\|(\nabla_u - \nabla_{u'})q'_\lambda\big\|_{W_2^{l,l/2}(Q(\frac{\lambda}{2}))} \\
&\quad = \big\|(\mathcal{A}_u - \mathcal{A}_{u'})\nabla q'_\lambda\big\|_{W_2^{l,l/2}(Q(\frac{\lambda}{2}))} \\
&\quad \leqslant C\big(\|\mathcal{A}_u - \mathcal{A}_{u'}\|_{L^\infty(t_0+\lambda/2,1;W_2^{l+1-2\epsilon}(\Omega))} + \|\mathcal{A}_u - \mathcal{A}_{u'}\|_{L^\infty(\Omega;W_2^{l/2+1/2-\epsilon}(t_0+\lambda/2,1))}\big) \\
&\qquad \times \big\|\nabla q'_\lambda\big\|_{W_2^{l,l/2}(Q(\frac{\lambda}{2}))} \\
&\quad \leqslant C\lambda^{-(l/2+\bar{\delta})}\|\mathcal{A}_u - \mathcal{A}_{u'}\|_{W_2^{l/2+1/2-\epsilon}(t_0+\lambda/2,1;W_2^{l+1-2\epsilon}(\Omega))}\|\nabla q\|_{W_2^{l,l/2}(Q(\frac{\lambda}{4}))},
\end{aligned} \tag{5.23}$$

where $\epsilon\in(0,l/2-1/4)$. It is easy to get

$$\|\mathcal{A}_u - \mathcal{A}_{u'}\|_{W_2^{l+1-2\epsilon}(\Omega)} = \left\|\int_{t-s}^{t}\frac{\partial\mathcal{A}_u}{\partial\tau}\,d\tau\right\|_{W_2^{l+1-2\epsilon}(\Omega)} \leqslant s^{\frac{1}{p'}}\left\|\frac{\partial\mathcal{A}_u}{\partial t}\right\|_{L^p(t_0+\lambda/2,1;W_2^{l+1-2\epsilon}(\Omega))},$$

where $\frac{1}{p}+\frac{1}{p'}=1$. Notice that $\frac{\partial\mathcal{A}_u}{\partial t}\in W_2^{l+1,l/2+1/2}(Q_1)$, we get by taking $\frac{1}{p}=\frac{1}{2}-\epsilon$, $\frac{1}{p'}=\frac{1}{2}+\epsilon$ and using Lemma 4.2 that

$$\|\mathcal{A}_u - \mathcal{A}_{u'}\|_{W_2^{l+1-2\epsilon}(\Omega)} \leqslant Cs^{\frac{1}{2}+\epsilon}\left\|\frac{\partial \mathcal{A}_u}{\partial t}\right\|_{W_2^{l+1,l/2+1/2}(Q(\frac{\lambda}{2}))} \leqslant Cs^{\frac{1}{2}+\epsilon},$$

which together with (5.23) implies that

$$\left\|(\nabla_u - \nabla_{u'})q'_\lambda\right\|_{W_2^{l,l/2}(Q(\frac{\lambda}{2}))} \leqslant Cs^{\frac{1}{2}+\epsilon}\lambda^{-(l/2+\bar{\delta})}\|\nabla q\|_{W_2^{l,l/2}(Q(\frac{\lambda}{4}))}.$$

Thanks to Lemmas 4.1–4.3, (5.3) and (5.8), we can estimate the remain terms in (5.21) similarly as above and finally we arrive at (5.22) by taking $\delta \ll 1$ and using (5.9). From (5.22), we obtain (5.10) by taking $\lambda = t_1 - t_0$. This completes the proof of the lemma. □

5.3. *Proof of Theorem 1.2*

The proof of Theorem 1.2 is based on a standard continuation argument of the local solution.

First of all, Theorem 1.1 implies that there exists a positive number ϵ_0 such that if $E_0 < \epsilon_0$, then the problem (1.8)–(1.9) is solvable on the interval [0, 1] and there holds

$$E(0,1) \leqslant C_1 E_0. \tag{5.24}$$

Next, we can easily see that conditions (5.3) and (5.8) are satisfied for sufficiently small E_0. In fact, from (5.24), we have

$$\left\|u^{(j)}\right\|_{W_2^{l+2,l/2+1}(Q_1^{(j)})} \leqslant C_1 E_0 < \delta,$$

and

$$\begin{aligned}
&\left\|F^{(j)} - h^{(j)}\right\|_{W_2^{l+3/2}(\mathbb{R}^2)} \\
&\quad\leqslant \left\|F^{(j)}\big(X_{u^{(j)}}(\xi,t),t\big) - F_0^{(j)}(\xi)\right\|_{W_2^{l+3/2}(S_F^{(j)})} + \left\|F_0^{(j)} - h^{(j)}\right\|_{W_2^{l+3/2}(\mathbb{R}^2)} \\
&\quad= \left\|\int_0^t u_3^{(j)}\,d\tau\right\|_{W_2^{l+3/2}(S_F^{(j)})} + \left\|F_0^{(j)} - h^{(j)}\right\|_{W_2^{l+3/2}(\mathbb{R}^2)} \\
&\quad\leqslant t^{1/2}\left\|u^{(j)}\right\|_{W_2^{l+2,0}(Q_t^{(j)})} + \left\|F_0^{(j)} - h^{(j)}\right\|_{W_2^{l+3/2}(\mathbb{R}^2)} \\
&\quad\leqslant C_1 E_0 + E_0 = (C_1+1)E_0 \leqslant \delta,
\end{aligned}$$

where we have used $(\frac{D}{Dt})^{(j)}(x_3 - F^{(j)}) = 0$.

So by Lemma 5.3 and (5.5), we can get

$$\begin{aligned}
&\sum_{j=1}^{2}\left(\left\|u^{(j)}\right\|_{W_2^{l+2,l/2+1}(Q^{(j)}(\lambda))} + \left\|\nabla q^{(j)}\right\|_{W_2^{l,l/2}(Q^{(j)}(\lambda))}\right) \\
&\quad + \left\|q^{(1)}\right\|_{W_2^{l+1/2,l/2+1/4}(S_F^{(1)}(\lambda))} + \left\|q^{(1)} - q^{(2)}\right\|_{W_2^{l+1/2,l/2+1/4}(S_F^{(2)}(\lambda))} \\
&\quad + \sup_{t_1<t<1}\sum_{j=1}^{2}\left(\left\|u^{(j)}\right\|_{W_2^{l+2}(\Omega^{(j)})} + \left\|q^{(j)}\right\|_{W_2^{l+1}(\Omega^{(j)})}\right)
\end{aligned}$$

$$\leqslant C_2 \sum_{j=1}^{2} \left(\| u^{(j)} \|_{L^2(Q(0))} + \sigma^{(j)} \| F^{(j)} - h^{(j)} \|_{L^2(\mathbb{R}^2 \times (0,1))} \right)$$
$$\leqslant C_2 E_0. \tag{5.25}$$

Finally, the boundary conditions, Lemma 5.2(ii) and (5.25) imply that

$$\begin{aligned}
\sigma^{(j)} \| F^{(j)} - h^{(j)} \|_{W_2^{l+5/2}(\mathbb{R}^2)} &\leqslant c_3 \| \sigma^{(j)} H^{(j)} \|_{W_2^{l+1/2}(\mathbb{R}^2)} + c_4 \sigma^{(j)} \| F^{(j)} - h^{(j)} \|_{L^2(\mathbb{R}^2)} \\
&\leqslant c_5 \sum_{j=1}^{2} \left(\| u^{(j)} \|_{W_2^{l+2}(\Omega^{(j)})} + \| q^{(j)} \|_{W_2^{l+1}(\Omega^{(j)})} \right) + c_4 \sigma^{(j)} \| F^{(j)} - h^{(j)} \|_{L^2(\mathbb{R}^2)} \\
&\leqslant C_3 E_0,
\end{aligned}$$

consequently,

$$\begin{aligned}
E_1 = \sup_{t_1 < t < 1} \sum_{j=1}^{2} \left(\| u^{(j)} \|_{W_2^{l+2}(\Omega^{(j)})} + \sigma^{(j)} \| F^{(j)} - h^{(j)} \|_{W_2^{l+5/2}(\mathbb{R}^2)} + \| \nabla \rho_0^{(j)} \|_{W_2^{l}(\mathbb{R}^3)} \right) \\
\leqslant C_4 E_0.
\end{aligned} \tag{5.26}$$

In particular, (5.26) is true for $t = 1$. Next, we consider the initial–boundary-value problem (1.1)–(1.2) at initial moment $t = 1$. Transform Lagrange coordinate system (ξ, t) with $\xi \in \Omega(1) = \Omega^{(1)}(1) \cap \Omega^{(2)}(1)$, $t > 1$, and construct the local solution. If we take ϵ that appears in Theorem 1.2 as

$$\epsilon = \min\left\{ \epsilon_0, \frac{\epsilon_0}{C_4}, \frac{\delta}{(C_1 + 1) C_4} \right\},$$

then we can repeat above argument again and again to obtain the global solution. This completes the proof of Theorem 1.2.

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